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THE HALTING PROBLEM RELATIVIZED TO COMPLEMENTS

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ABSTRACT. Let $H^A=\{e|\text{domain}\{e\}\cap A\neq\varnothing\}$. It is shown that there exists a set A of Turing degree $\boldsymbol{a}$ such that H^A is Turing-incomparable to $H^{\bar{A}}$ whenever $\boldsymbol{a}$ is an r.e. degree with $\boldsymbol{a}'>\mathbf{0}'$, or $\boldsymbol{a}\geqq\mathbf{0}''$ or $\boldsymbol{a}\geqq\mathbf{0}'$ and $\boldsymbol{a}$ is r.e. in $\mathbf{0}'$. This contrasts with the fact that H^A is comparable to $H^{\bar{A}}$ for almost all A.

1. **Introduction.** In [7], the "relativized halting problem" is defined to be the problem of deciding, for a fixed set A, whether domain $\{e\}\cap A\neq\varnothing$, where $\{e\}$ denotes the eth partial recursive function. It is shown in that paper that the difficulty of the halting problem relativized to A is not a simple function of the "complexity" of A, in the sense that the Turing degree of $H^A=\{e|\text{domain}\{e\}\cap A\neq\varnothing\}$ is, in general, independent of the Turing degree of A; the same result was obtained independently in [2] for the special case where A is r.e., nonrecursive. (The context of [2] was not given as a "halting problem", but as the classification of $\theta A^B=\{e|W_e\subset B\}$ where $W_e=\text{domain}\{e\}$; but evidently $\theta A^B=\bar{H}^{\bar{B}}$, where $\bar{B}$ denotes the complement of B.) In both [2] and [7] it is shown that it is possible for H^A and $H^{\bar{A}}$ to have *different* Turing degrees, and in [7] the author conjectures that there exists a set A such that H^A and $H^{\bar{A}}$ have *incomparable* Turing degrees. The purpose of this paper is to prove this conjecture true; such sets occur, in fact, in all Turing degrees $\boldsymbol{a}$ satisfying one of the following conditions:

(i) $\boldsymbol{a}$ is an r.e. degree with $\boldsymbol{a}'>\mathbf{0}'$;
(ii) $\boldsymbol{a}\geqq\mathbf{0}''$;
(iii) $\boldsymbol{a}\geqq\mathbf{0}'$ and $\boldsymbol{a}$ is r.e. in $\mathbf{0}'$.

In the concluding section we discuss why some such restrictions are necessary.

2. **Main results.** The notation is that of [5], except that $\{2x|x\in A\}\cup\{2x+1|x\in B\}$ will be denoted by $A\oplus B$. The Turing degree of A will be denoted by $d(A)$; in this notation, $d(A\oplus B)=d(A)\cup d(B)$. Following

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[3], a set A is called *bi-immune* if A and $\bar{A}$ are immune, and a degree is called bi-immune if it contains a bi-immune set. A set A is called *d.r.e.* if there exist r.e. sets B and C with $A=B-C$. For any set A, $\tilde{A}$ will denote $\{\langle u, v\rangle | D_u \subset A \& D_v \subset \bar{A}\}$, where $\{D_i\}$ is the canonical indexing of finite sets. The techniques used in our proofs are based on those of [7], but some additional work is needed to control complements.

THEOREM 1. *Let $\boldsymbol{a}$ be an r.e. degree with $\boldsymbol{a}' > \boldsymbol{0}'$. Then there exists a set A satisfying*

(i) $d(A)=\boldsymbol{a}$,

(ii) $d(H^A)|d(H^{\bar{A}})$,

(iii) *A and $\bar{A}$ are d.r.e.*

We first list some known results which will be needed for the proof.

Fact 1 (Robinson [4]). If $\boldsymbol{a} \leqq \boldsymbol{b} < \boldsymbol{c}$ and $\boldsymbol{b}$, $\boldsymbol{c}$ are r.e. in $\boldsymbol{a}$, then there exist degrees $\boldsymbol{b}_0$, $\boldsymbol{b}_1$ which are r.e. in $\boldsymbol{a}$ and such that $\boldsymbol{b} < \boldsymbol{b}_0$, $\boldsymbol{b}_1 < \boldsymbol{c}$ and $\boldsymbol{b}_0 | \boldsymbol{b}_1$.

Fact 2 (Hay [2]). If $\boldsymbol{a}$ is r.e., nonrecursive, and $\boldsymbol{b}$ is a degree r.e. in $\boldsymbol{a}$ with $\boldsymbol{b} \geqq \boldsymbol{0}'$, then there exists an r.e. set A such that $d(A)=\boldsymbol{a}$ and $d(H^{\bar{A}})=\boldsymbol{b}$.

Fact 3 (Selman [7]). For all sets A, B, $d(H^{A \oplus B})=d(H^A) \cup d(H^B)$.

LEMMA 1. *If A is r.e. and nonempty, then $d(H^A)=\boldsymbol{0}'$.*

PROOF. If A is r.e., H^A is evidently also r.e., so $d(H^A) \leqq \boldsymbol{0}'$. Conversely, $A \neq \varnothing$ implies $H^A \neq \varnothing$, so $\boldsymbol{0}' \leqq d(H^A)$ by Rice's theorem [5].

LEMMA 2. *If A and B are r.e., then $A \oplus \bar{B}$ and $(A \oplus \bar{B})^-$ are d.r.e.*

PROOF. Let $E=\{x | x \text{ is even}\}$, $A_0=\{x | [x/2] \in A\}$, $B_0=\{x | [x/2] \in B\}$. Then E, $\bar{E}$, A_0, B_0 are clearly r.e., and $A \oplus \bar{B}=[(E \cap A_0) \cup \bar{E}]-B_0$, $(A \oplus \bar{B})^-=\bar{A} \oplus B=[E \cup (\bar{E} \cap B_0)]-A_0$.

PROOF OF THEOREM 1. Suppose $\boldsymbol{a}$ is r.e. and $\boldsymbol{a}' > \boldsymbol{0}'$. Then $\boldsymbol{a} \leqq \boldsymbol{0}' < \boldsymbol{a}'$, with $\boldsymbol{0}'$ and $\boldsymbol{a}'$ both r.e. in $\boldsymbol{a}$. Hence by Fact 1, there exist degrees $\boldsymbol{b}_0$, $\boldsymbol{b}_1$ r.e. in $\boldsymbol{a}$ with $\boldsymbol{0}' \leqq \boldsymbol{b}_0$, $\boldsymbol{b}_1$ and $\boldsymbol{b}_0 | \boldsymbol{b}_1$. By Fact 2, there exist r.e. sets A_0, A_1 with $d(A_0)=d(A_1)=\boldsymbol{a}$, $d(H^{\bar{A}_0})=\boldsymbol{b}_0$ and $d(H^{\bar{A}_1})=\boldsymbol{b}_1$. Now $\boldsymbol{a}' > \boldsymbol{0}'$ implies $\boldsymbol{a} > \boldsymbol{0}$, hence A_0 and A_1 are nonempty; so by Lemma 1, $d(H^{A_0})=d(H^{A_1})=\boldsymbol{0}'$. Let $A=A_0 \oplus (A_1)^-$; clearly $d(A)=\boldsymbol{a}$ and, by Fact 3, $d(H^A)=d(H^{A_0 \oplus \bar{A}_1})= d(H^{A_0}) \cup d(H^{\bar{A}_1})=\boldsymbol{0}' \cup \boldsymbol{b}_1=\boldsymbol{b}_1$. Similarly, since $\bar{A}=\bar{A}_0 \oplus A_1, d(H^{\bar{A}})=d(H^{\bar{A}_0}) \cup d(H^{A_1})=\boldsymbol{b}_0 \cup \boldsymbol{0}'=\boldsymbol{b}_0$. So $d(H^A)|d(H^{\bar{A}})$ and, by Lemma 2, A and $\bar{A}$ are d.r.e., which completes the proof.

Note that in Theorem 1, "d.r.e." cannot be improved to "r.e." even for one of the sets A, $\bar{A}$, since if A is r.e. and neither A nor $\bar{A}$ is empty, then $d(H^A)=\boldsymbol{0}' \leqq d(H^{\bar{A}})$ by Lemma 1 and Rice's theorem.

THEOREM 2. *Assume $\boldsymbol{0}' \leqq \boldsymbol{a} \leqq \boldsymbol{b}_0$, $\boldsymbol{b}_1$. If there exist bi-immune degrees $\boldsymbol{c}_0$, $\boldsymbol{c}_1$ such that $\boldsymbol{c}_0$, $\boldsymbol{c}_1 \leqq \boldsymbol{a}$, $\boldsymbol{b}_0=\boldsymbol{c}_0'$ and $\boldsymbol{b}_1=\boldsymbol{c}_1'$, then there is a set A satisfying $d(A)=\boldsymbol{a}$, $d(H^A)=\boldsymbol{b}_0$, $d(H^{\bar{A}})=\boldsymbol{b}_1$.*

We again list some known results needed for the proof.

Fact 4 (Selman [7]). Let $d(B)=\boldsymbol{b}$. If $\bar{B}$ is immune, then $d(H^{\bar{B}})=\boldsymbol{b}\cup\boldsymbol{0}'$.

Fact 5 [5, Exercise 9-31]. For all $\boldsymbol{a}$, if $\boldsymbol{a}\geqq\boldsymbol{0}'$ then $\boldsymbol{a}$ is bi-immune.

Fact 6 [5, Exercise 9-50]. If $\tilde{A}=\{\langle u, v\rangle \mid D_u\subset A \,\&\, D_v\subset \bar{A}\}$, then $d(H^{\tilde{A}})=\boldsymbol{a}'$.

LEMMA 3. *For all sets A, if A is bi-immune then $\bar{\tilde{A}}$ is immune.*

PROOF. Assume A and $\bar{A}$ are both immune. In particular, they are then infinite, so they have infinitely many finite subsets and hence $\tilde{A}$ is infinite. Let W be an r.e. subset of $\tilde{A}$. Define

$$W_0 = \{u \mid (\exists v)(\langle u, v\rangle \in W)\}, \qquad W_1 = \{v \mid (\exists u)(\langle u, v\rangle \in W)\}.$$

Then W_0 and W_1 are r.e., and, for all u, v, $u \in W_0$ implies $D_u\subset A$, $v \in W_1$ implies $D_v\subset \bar{A}$. Suppose W_0 is infinite, and let

$$V_0 = \{x \mid (\exists u)(u \in W_0 \,\&\, x \in D_u)\}.$$

Then V_0 is infinite, since the canonical indexing is one-one and hence V_0 is the union of infinitely many different finite sets. But then V_0 is an infinite r.e. subset of A, which contradicts the assumption that A is immune. So W_0 is finite, and by a symmetric argument using W_1 and $\bar{A}$, W_1 is also finite. But then $W_0\times W_1$ is finite, and hence so is W since $W\subset W_0\times W_1$. So every r.e. subset of $\tilde{A}$ is finite, and $\tilde{A}$ is immune.

LEMMA 4. *Let $\boldsymbol{a}$ be bi-immune. Then there is a set B such that $d(B)=\boldsymbol{a}$, $d(H^B)=\boldsymbol{a}'$ and $d(H^{\bar{B}})=\boldsymbol{a}\cup\boldsymbol{0}'$.*

PROOF. Assume $\boldsymbol{a}$ contains a bi-immune set A, and let $B=\tilde{A}$; clearly $d(B)=d(A)=\boldsymbol{a}$. By Fact 6, $d(H^B)=d(H^{\tilde{A}})=\boldsymbol{a}'$. By Lemma 3, $\bar{B}$ is immune, so $d(H^{\bar{B}})=d(B)\cup\boldsymbol{0}'=\boldsymbol{a}\cup\boldsymbol{0}'$, by Fact 4.

PROOF OF THEOREM 2. Assume $\boldsymbol{0}'\leqq\boldsymbol{a}\leqq\boldsymbol{b}_0, \boldsymbol{b}_1$, and that there exist bi-immune degrees $\boldsymbol{c}_0, \boldsymbol{c}_1$ satisfying $\boldsymbol{c}_0, \boldsymbol{c}_1\leqq\boldsymbol{a}$, $\boldsymbol{b}_0=\boldsymbol{c}_0'$, $\boldsymbol{b}_1=\boldsymbol{c}_1'$. By Lemma 4, there exist sets C_0, C_1 such that for $i=0, 1$, $d(C_i)=\boldsymbol{c}_i$, $d(H^{C_i})=\boldsymbol{c}_i'=\boldsymbol{b}_i$ and $d(H^{\bar{C}_i})=\boldsymbol{c}_i\cup\boldsymbol{0}'$. By Fact 5, $\boldsymbol{0}'\leqq\boldsymbol{a}$ implies $\boldsymbol{a}$ is bi-immune. Let A_0 be any bi-immune set such that $d(A_0)=\boldsymbol{a}$; then by Fact 4,

$$d(H^{A_0}) = d(H^{\bar{A}_0}) = \boldsymbol{a} \cup \boldsymbol{0}' = \boldsymbol{a}.$$

Let $A=(C_0\oplus\bar{C}_1)\oplus A_0$; then $\bar{A}=\overline{(C_0\oplus\bar{C}_1)}\oplus\bar{A}_0=(\bar{C}_0\oplus C_1)\oplus\bar{A}_0$, and

$$d(A) = d(C_0) \cup d(C_1) \cup d(A_0) = \boldsymbol{c}_0 \cup \boldsymbol{c}_1 \cup \boldsymbol{a} = \boldsymbol{a}$$

since $\boldsymbol{c}_0, \boldsymbol{c}_1\leqq\boldsymbol{a}$. By Fact 3,

$$d(H^A) = d(H^{C_0}) \cup d(H^{\bar{C}_1}) \cup d(H^{A_0}) = \boldsymbol{b}_0 \cup \boldsymbol{c}_1 \cup \boldsymbol{0}' \cup \boldsymbol{a} = \boldsymbol{b}_0$$

since $\boldsymbol{c}_1\cup\boldsymbol{0}'\leqq\boldsymbol{a}\leqq\boldsymbol{b}_0$, and, similarly,

$$d(H^{\bar{A}}) = d(H^{\bar{C}_0}) \cup d(H^{C_1}) \cup d(H^{\bar{A}_0}) = \boldsymbol{c}_0 \cup \boldsymbol{0}' \cup \boldsymbol{b}_1 \cup \boldsymbol{a} = \boldsymbol{b}_1.$$

So A satisfies the conclusion of the theorem.

THEOREM 3. *Let $\boldsymbol{a}$ be any degree such that $\boldsymbol{a} \geqq \boldsymbol{0}''$. Then there exists a set A such that $d(A)=\boldsymbol{a}$ and $d(H^A)|d(H^{\bar{A}})$.*

The proof will be an application of Theorem 2. We list some additional facts which will be needed.

Fact 7 (Selman [6]). For all $\boldsymbol{a}$, if $\boldsymbol{a} \geqq \boldsymbol{0}''$ then there exists a degree $\boldsymbol{c}$ such that $\boldsymbol{a}=\boldsymbol{c}''$.

Fact 8 (Shoenfield [8]). Let $\boldsymbol{b}$, $\boldsymbol{c}$ be degrees such that $\boldsymbol{c}' \leqq \boldsymbol{b}$ and $\boldsymbol{b}$ is r.e. in $\boldsymbol{c}'$. Then there is a degree $\boldsymbol{d}$ with $\boldsymbol{c} \leqq \boldsymbol{d} \leqq \boldsymbol{c}'$ and $\boldsymbol{d}'=\boldsymbol{b}$.

PROOF OF THEOREM 3. Assume $\boldsymbol{a} \geqq \boldsymbol{0}''$. By Fact 7, $\boldsymbol{a}=\boldsymbol{c}''$ for some degree $\boldsymbol{c}$. Apply Fact 1 to obtain degrees $\boldsymbol{b}_0$, $\boldsymbol{b}_1$ r.e. in $\boldsymbol{a}$ with $\boldsymbol{a} \leqq \boldsymbol{b}_0, \boldsymbol{b}_1 \leqq \boldsymbol{a}'$ and $\boldsymbol{b}_0|\boldsymbol{b}_1$. (This special case of Fact 1 is of course just the Friedberg-Muchnik theorem relativized to $\boldsymbol{a}$.) Then by Fact 8, there exist degrees $\boldsymbol{c}_0$, $\boldsymbol{c}_1$ with $\boldsymbol{b}_0=\boldsymbol{c}_0'$, $\boldsymbol{b}_1=\boldsymbol{c}_1'$ and $\boldsymbol{0}' \leqq \boldsymbol{c}' \leqq \boldsymbol{c}_0, \boldsymbol{c}_1 \leqq \boldsymbol{c}''=\boldsymbol{a}$. By Fact 5, $\boldsymbol{0}' \leqq \boldsymbol{c}_0, \boldsymbol{c}_1$ implies $\boldsymbol{c}_0$ and $\boldsymbol{c}_1$ are bi-immune. The hypotheses of Theorem 2 therefore apply to $\boldsymbol{a}$, $\boldsymbol{b}_0$, $\boldsymbol{b}_1$; hence there is a set A such that $d(A)=\boldsymbol{a}$ and $d(H^A)=\boldsymbol{b}_0|\boldsymbol{b}_1=d(H^{\bar{A}})$.

THEOREM 4. *If $\boldsymbol{a} \geqq \boldsymbol{0}'$ and $\boldsymbol{a}$ is r.e. in $\boldsymbol{0}'$, then there exists a set A such that $d(A)=\boldsymbol{a}$ and $d(H^A)|d(H^{\bar{A}})$.*

We list one more fact needed to apply Theorem 2.

Fact 9 (Jockusch [3]). If $\boldsymbol{d}$ has a nonzero predecessor $\boldsymbol{c}$ which is r.e. in $\boldsymbol{0}'$, then $\boldsymbol{d}$ is bi-immune.

PROOF OF THEOREM 4. Assume $\boldsymbol{a} \geqq \boldsymbol{0}'$ and $\boldsymbol{a}$ is r.e. in $\boldsymbol{0}'$. Then by Fact 8, there exists a degree $\boldsymbol{c} \leqq \boldsymbol{0}'$ such that $\boldsymbol{a}=\boldsymbol{c}'$. Again by Fact 1, there exist degrees $\boldsymbol{b}_0$, $\boldsymbol{b}_1$ r.e. in $\boldsymbol{a}$ with $\boldsymbol{a}<\boldsymbol{b}_0, \boldsymbol{b}_1$ and $\boldsymbol{b}_0|\boldsymbol{b}_1$. Apply Fact 8 again to obtain degrees $\boldsymbol{c}_0$, $\boldsymbol{c}_1$ with $\boldsymbol{b}_0=\boldsymbol{c}_0'$, $\boldsymbol{b}_1=\boldsymbol{c}_1'$ and $\boldsymbol{c} \leqq \boldsymbol{c}_0, \boldsymbol{c}_1 \leqq \boldsymbol{c}'=\boldsymbol{a}$. But $\boldsymbol{c} \leqq \boldsymbol{0}'$ implies $\boldsymbol{c}$ is r.e. in $\boldsymbol{0}'$; hence by Fact 9, $\boldsymbol{c}_0$, $\boldsymbol{c}_1$ are bi-immune. Thus the hypotheses of Theorem 2 again apply to $\boldsymbol{a}$, $\boldsymbol{b}_0$, $\boldsymbol{b}_1$; hence there is a set A satisfying $d(A)=\boldsymbol{a}$ and $d(H^A)=\boldsymbol{b}_0|\boldsymbol{b}_1=d(H^{\bar{A}})$.

3. **Concluding remarks.** To what extent are the conditions on $\boldsymbol{a}$ in the hypotheses of Theorems 1, 3 and 4 necessary? In [7] it is pointed out that membership information concerning H^A gives no information regarding membership in $H^{\bar{A}}$—and certainly in the case where W_e is infinite it is not clear how one might hope to decide (in general) whether $W_e \cap \bar{A} \neq \varnothing$ on the basis of finitely many questions of the type: Does $W_x \cap A=\varnothing$? It is tempting to conclude that $d(H^A)|d(H^{\bar{A}})$ must be the rule rather than the exception; but, somewhat curiously, this is not the case. For purposes of this discussion, call a set A *free* if $d(H^A)$ is incomparable to $d(H^{\bar{A}})$, *bound* otherwise. Call a degree $\boldsymbol{a}$ *totally free* if every set in $\boldsymbol{a}$ is free, and

totally bound if every set in $\boldsymbol{a}$ is bound. The following then hold:

(a) Almost all sets A are bound.

(b) No degree is totally free.

(c) Almost all degrees are totally bound.

Here "almost all" means "on a set of measure 1" in the usual measure on Baire space and the induced measure on the set of all Turing degrees (as described, e.g., in [9]). To verify (a)–(c), note first that

(d) (Selman [7].) For all sets A, if $d(A)=\boldsymbol{a}$ then $\boldsymbol{a}\cup\boldsymbol{0}'\leqq d(H^A)\leqq\boldsymbol{a}'$. Hence A is bound whenever $d(H^{\bar{A}})=\boldsymbol{a}\cup\boldsymbol{0}'$; since the latter holds, by Fact 4, whenever A is immune, it follows that

(e) Every immune set is bound.

But it is a well-known (and easily verified) fact that almost every set is immune, hence (a) holds. Also, it was proved in [1] that every nonzero degree contains an immune set; so (b) holds for nonzero degrees (for $\boldsymbol{0}$, it holds by Lemma 1). Finally it is clear from (d) that a degree $\boldsymbol{a}$ is totally bound if $\boldsymbol{a}'=\boldsymbol{a}\cup\boldsymbol{0}'$. But the latter holds for almost all $\boldsymbol{a}$, by a result attributed to Sacks in [9], which proves (c). In the present terminology, Theorems 1, 3 and 4 of the previous section give conditions under which a degree $\boldsymbol{a}$ is not totally bound; in light of (a)–(c) it is clear that some strong restrictions on $\boldsymbol{a}$ are in fact necessary.

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