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DESCENT OF PROJECTIVITY FOR LOCALLY FREE MODULES

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ABSTRACT. Let $R\rightarrow\hat{R}$ be the natural homomorphism from the commutative ring R into its associated von Neumann regular ring, and let M be a locally free R-module such that $\hat{R}\otimes M$ is a projective $\hat{R}$-module. We show that if M is either countably generated or locally finitely generated, then M is projective, and we deduce that the trace of any projective ideal is projective. These results are a consequence of a more general theorem on the descent of the Mittag-Leffler condition. The "locally free" hypothesis may be weakened to "flat" if and only if R is locally perfect.

1. **Introduction.** Let $\phi:R\rightarrow R'$ be a ring homomorphism and M an R-module such that $R'\otimes M$ is projective (as an R'-module). The general problem of "descent of projectivity" is to find conditions on M and ϕ that force M to be projective. The most definitive results have traditionally required M to be finitely generated and flat. Recently, however, Raynaud and Gruson have used a technique of "approximation by finitely generated modules" to obtain some remarkable theorems on descent of projectivity for nonfinitely generated modules **[RG]**.

In the present paper we use these techniques to study the natural homomorphism $\phi:R\rightarrow\hat{R}$, where R is a commutative ring and $\hat{R}$ is its associated von Neumann regular ring **[O]**. It was shown in **[W1]** that if M is finitely generated and flat, and $\hat{R}\otimes M$ is projective, then M is projective. Unfortunately, the hypothesis of finite generation can be dropped only for rings that are locally perfect. It seems profitable to replace flatness by the stronger assumption that M be locally free. In this case, finiteness of M may be replaced by either "countably generated" or "locally finitely generated". These results are easy consequences of our main theorem on the descent of the Mittag-Leffler condition.

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All rings considered are commutative with identity. An R-module M has a property "locally" if M_P has that property as an R_P-module for each maximal ideal P. If $R \to R'$ is a ring homomorphism, the statement "$R' \otimes M$ has property x" means "$R' \otimes_R M$ has property x as an R'-module". If s is an element of a ring R, the ring of fractions $\{(r/s^n)\}$ is denoted by R_s.

2. **The main theorem.** We refer the reader to the paper by Raynaud and Gruson [**RG**] for a thorough treatment of Mittag-Leffler modules. For convenience, we recall the pertinent definitions and results.

Let $f: A \to B$ and $g: A \to C$ be homomorphisms of R-modules. Then g is said to *dominate* f provided $\mathrm{Ker}(1_X \otimes f) \subseteq \mathrm{Ker}(1_X \otimes g)$ for every R-module X. We remark that if g factors through f then g dominates f, and that domination is a transitive relation on maps emanating from A. Suppose F is finitely presented and $u: F \to M$. A map $v: F \to G$ is called a *stabilizer* for u provided G is finitely presented and u and v dominate each other. An R-module M is *Mittag-Leffler* if every map from every finitely presented module into M admits a stabilizer. Examples of Mittag-Leffler modules include all pure-projective modules and their pure submodules. We shall need the following characterization of Mittag-Leffler modules, which is easily deduced from the proof of [**RG**, II, Proposition 2.1.4] and the remarks following the definition of "dominate" given above:

LEMMA 2.1. *Let (F_i, u_{ji}) be a direct system of finitely presented modules with direct limit (M, u_i).*

(1) *If some $u_i: F_i \to M$ admits a stabilizer, then there is an index $j_0 \geqq i$ such that u_{ji} stabilizes u_i whenever $j \geqq j_0$.*

(2) *If every u_i admits a stabilizer then M is Mittag-Leffler.*

Every ring R admits a homomorphism $\phi: R \to \hat{R}$ characterized by the following universal property: (1) $\hat{R}$ is (von Neumann) regular, and (2) every homomorphism from R into a regular ring factors uniquely through ϕ. (This was observed independently by J.-P. Olivier [**O**] and M. Hochster [**H**].) The induced map ${}^a\phi$:spec$(\hat{R}) \to$ spec(R) is a bijection. For each $P \in \mathrm{spec}(R)$, ϕ induces an isomorphism between the fields R_P/PR_P and $\hat{R}_{\hat{P}}$, where $\hat{P}$ is the unique prime (=maximal) ideal of $\hat{R}$ such that ${}^a\phi(\hat{P}) = P$. If spec(R) is retopologized so as to make ${}^a\phi$ a homeomorphism, the result is the (stronger) patch topology [**H**]. Proofs of these assertions may be found in [**W2**]. We now come to the main theorem.

THEOREM 2.2. *Let M be a flat R-module. Then M is Mittag-Leffler if and only if M is locally Mittag-Leffler and $\hat{R} \otimes M$ is Mittag-Leffler.*

PROOF. Write M as the direct limit of the direct system (F_i, u_{ji}), with each F_i finitely presented [L, Appendice]. The "only if" implication is a direct consequence of Lemma 2.1. To prove the converse, fix an index i. We need an index $j \geqq i$ such that u_{ji} stabilizes $u_i: F_i \to M$.

Let P be a prime ideal, fixed for the moment. Since M is locally Mittag-Leffler, Lemma 2.1 implies that M_P is a Mittag-Leffler R_P-module. (It is necessary to work with *all* primes—not just the maximal ones—since we will eventually need a topological condition that depends on the existence of generic points.) By [RG, II, 2.2.1], $\mathrm{Im}((u_i)_P)$ is contained in a countably generated, pure-projective, pure R_P-submodule F of M_P. Since F is flat, it is projective, and hence free. Then $\mathrm{Im}((u_i)_P)$ is contained in a finitely generated free summand of F. It follows that there exist a free R-module L of finite rank and a map $w: L \to M$ such that $w_P: L_P \to M_P$ is a pure monomorphism and $\mathrm{Im}((u_i)_P) \subseteq \mathrm{Im}(w_P)$. (At this point we are reproducing part of the proof of [RG, II, 2.5.6], since that result appears to have a misprint in its statement as well as a minor obscurity in its proof.)

Let $U = \{Q \in \mathrm{spec}(R) | \mathrm{Im}((u_i)_Q) \subseteq \mathrm{Im}(w_Q)\}$ and $V = \{Q \in \mathrm{spec}(R) | w_Q$ is a pure monomorphism$\}$. We claim $U \cap V$ is a neighborhood of P. Assuming this for the time being, we can easily complete the proof as follows: Choose an element $s \in R$ such that $P \in D(s) \subseteq U \cap V$, where $D(s)$ is the set of primes not containing P. Identifying $\mathrm{spec}(R_s)$ with $D(s)$ and globalizing, we see that $w_s: L_s \to M_s$ is a pure monomorphism, and $\mathrm{Im}((u_i)_s) \subseteq \mathrm{Im}(w_s)$. Clearly, the map $(F_i)_s \to \mathrm{Im}(w_s)$ (induced by $(u_i)_s$) stabilizes $(u_i)_s$. By Lemma 2.1 there is an index $j_0 \geqq i$ such that $(u_{ji})_s$ stabilizes $(u_i)_s$ for each $j \geqq j_0$. By administering the same treatment to each $P \in \mathrm{spec}(R)$, we obtain, by compactness, an index $j \geqq i$ such that u_{ji} stabilizes u_i locally, and hence globally.

To prove our claim, we observe that U is open, since it is the complement of the support of the finitely generated module $(\mathrm{Im}(u_i) + \mathrm{Im}(w))/\mathrm{Im}(w)$. The set $\mathrm{spec}(R) - V$ clearly contains the closure of each of its points. By the first corollary to Theorem 1 of [H], such a set is closed in the Zariski topology if and only if it is closed in the patch topology. Thus it suffices to show that V is open in the patch topology.

Let K be the kernel of $\hat{w}: \hat{R} \otimes L \to \hat{R} \otimes M$. Since $\hat{R}$ is regular, $\mathrm{Im}(\hat{w})$ is a (finitely generated) pure submodule of the Mittag-Leffler $\hat{R}$-module $\hat{R} \otimes M$, and hence is projective [RG, II, 2.1.6, 2.2.2]. Therefore K is a finitely generated projective $\hat{R}$-module, and it follows that $\{Q \in \mathrm{spec}(R) | K_{\hat{Q}} = 0\}$ is open (and closed) in the patch topology. However, the next lemma shows that this set is precisely V, and hence completes the proof.

LEMMA 2.3 [RG, I, 3.1.6]. *Let R be a local ring with residue field k.*

Let $u: F \to M$ be an R-homomorphism with F free and M flat. Then u is a pure monomorphism if and only if $1_k \otimes u$ is a monomorphism.

The hypothesis that M be flat cannot be deleted from Theorem 2.2. For example, let R be a nonnoetherian ring such that spec(R) is noetherian, and R_P is noetherian for each maximal ideal P. (Such a ring is constructed in [**HO**].) Let M be a finitely generated module that is not finitely presented (and hence not Mittag-Leffler, by [**RG**, II, 2.2.2]). Clearly M is locally Mittag-Leffler, and by [**W2**, Theorem 2] $\hat{R} \otimes M$ is projective (and hence Mittag-Leffler).

Let j-spec(R) denote the set of prime ideals of R that are intersections of maximal ideals. As in [**W3**], let $\tilde{R} = \hat{R} / \bigcap \{\hat{P} \mid P \in j\text{-spec}(R)\}$. If j-spec(R) is closed in the patch topology on spec(R), we say R is *j-closed*. In this case the map $R \to \tilde{R}$ induces a bijection between spec($\tilde{R}$) and j-spec(R), and the proof of Theorem 2.2 can easily be modified to give the following result:

THEOREM 2.4. *Suppose R is j-closed and M is a flat R-module. Then M is Mittag-Leffler if and only if M is locally Mittag-Leffler and $\tilde{R} \otimes M$ is Mittag-Leffler.*

3. **Applications to the descent of projectivity.** According to [**RG**, II, 2.2.2], a countably generated flat module is Mittag-Leffler if and only if it is projective. The following is therefore an immediate consequence of Theorem 2.2:

PROPOSITION 3.1. *A countably generated R-module M is projective if and only if M is locally free and $\hat{R} \otimes M$ is projective.*

It is unknown to the author whether the countability hypothesis can be dropped, but it can be replaced by local finiteness.

PROPOSITION 3.2. *Let M be a locally finitely generated R-module. Then M is projective if and only if M is locally free and $\hat{R} \otimes M$ is projective.*

PROOF. Write $\hat{R} \otimes M = \bigoplus \sum Q_i$ $(i \in I)$, where each Q_i is a countably generated $\hat{R}$-module, and let x be an element of M. By Theorem 2.2, M is Mittag-Leffler, and the proof of [**RG**, II, 3.1.3] provides a countably generated, pure submodule M' of M such that $x \in M'$ and $\hat{R} \otimes M' = \bigoplus \sum Q_j$ $(j \in J)$ for some $J \subseteq I$. Then M' is locally free, and hence locally finitely generated, and it follows from purity that M/M' is locally free. Since $\hat{R} \otimes (M/M') = \bigoplus \sum Q_i$ $(i \in I - J)$, all our hypotheses on M carry over to M/M'. A transfinite induction argument, as in the proof of [**RG**, II, 3.1.3], completes the proof.

COROLLARY 3.3. *Let M be a projective R-module such that for each P, M_P is free of rank* 0 *or* 1 (*for example, take M to be a projective ideal of R*). *Then the trace ideal of M is projective.*

PROOF. Let T be the trace of M. Then T_P is free of rank 0 or 1 for each P, and $\hat{R}\otimes T$ is the trace of $\hat{R}\otimes M$ [V]. Now $\hat{R}\otimes M$ is isomorphic to a direct sum of principal ideals $\hat{R}e_\alpha$, $e_\alpha=e_\alpha^2 \in \hat{R}$, by [**K**]. Since $\hat{R}_{\hat{P}}\otimes M$ has rank 0 or 1 for each P, it follows easily that the idempotents e_α are orthogonal, and hence $\hat{R}\otimes M$ is isomorphic to an ideal I of $\hat{R}$. Let $e=e^2 \in I$ and let $f \in \mathrm{Hom}_{\hat{R}}(I, \hat{R})$. Then $f(e)=ef(e) \in I$. Since I is generated by its idempotents, it follows that I is its own trace. Thus $\hat{R}\otimes T=I$, and since I is projective, Proposition 3.2 implies that T is projective.

If one could get rid of the countability hypothesis in Proposition 3.1, the resulting theorem and its $\tilde{R}$-analogue would have many applications to homological dimension theory. For example, it would follow that the homological dimension of a flat module M is the maximum of $\sup_P$ h. $\dim_{R_P} M_P$ and h. $\dim_{\hat{R}}(\hat{R}\otimes M)$. The usefulness of results like this would stem from the fact that it is comparatively easy to get bounds on the global dimensions of $\hat{R}$ and $\tilde{R}$. For example, if j-spec(R) is a noetherian space of dimension d, then R is j-closed, and gl $\dim(\tilde{R})\leqq d$, by [**W3**, 1.1, 5.3] and [**W2**, Proposition 4].

4. **Flatness is not enough.** For which rings R can one replace "locally free" by "flat" in the hypotheses of Proposition 3.1? The class of rings in question is rather small, but provides a natural, common generalization of "regular" and "perfect".

PROPOSITION 4.1. *The following conditions on a ring R are equivalent*:

(i) (*resp.* (ii)) *If M is an arbitrary* (*resp. a countably generated*) *flat R-module, and $\hat{R}\otimes M$ is projective, then M is projective.*

(iii) *The nilradical of R is T-nilpotent, and* $\dim(R)=0$ (*that is, primes are maximal*).

(iv) *R_P is perfect for every maximal ideal P.*

PROOF. The program is to show that (iii)$\Rightarrow$(iv)$\Rightarrow$(ii) and (iii)$\Rightarrow$(i)$\Rightarrow$(ii)$\Rightarrow$(iii). Assume (iii) holds, let N be the nilradical of R, and let $\{x_k\}$ be a sequence of elements in a maximal ideal P. Since PR_P is a nilideal of R_P, there exist elements $s_k \in R-P$ such that $s_kx_k \in N$. Then, for some n, $s_1 \cdots s_n x_1 \cdots x_n=0$, and it follows that PR_P is T-nilpotent. Therefore R_P is perfect, and (iv) is verified. Since flats are projective over a perfect ring [**B**], (iv) implies (ii) by Proposition 3.1.

Suppose (iii) is satisfied. Then R/N is regular, since its localizations are 0-dimensional local rings with no nilpotents, that is, fields. Therefore $R/N=\hat{R}$, and since N is T-nilpotent, it follows that (i) is satisfied, by [**RG**, II, 1.2.6, 3.1.4(1)]. Obviously, (i) implies (ii), and the proof will be complete once we check that (ii) implies (iii).

Let $\{x_k\}$ be a sequence of elements of N. We need an n such that $x_1 \cdots x_n = 0$. As in [**RG**, II, 1.2.6], let M be the direct limit of the system $R \to^{x_1} R \to^{x_2} R \to \cdots$. Then M is a countably generated flat R-module, and $(R/N)\otimes M=0$. Since ϕ factors through the natural map $\pi: R \to R/N$, we see that $\hat{R}\otimes M=0$. By condition (ii), M is projective, and it follows from [**B**, 2.7] that $M=0$. Hence $x_1 \cdots x_n=0$ for a suitable n. To show that $\dim(R)=0$, let s be an arbitrary element of R. Then R_s is a countably generated, flat R-module, and $\hat{R}\otimes R_s$ is projective by [**W1**, Corollary 2]. By (ii) R_s is projective, and a typical "dual basis" argument (e.g. [**CE**, p. 132]) shows that R_s is finitely generated, and hence cyclic. Therefore, for some n, we have $Rs^n = Rs^{n+1}$. The following observation completes the proof:

PROPOSITION 4.2. *Let R be any ring. Then* $\dim(R)=0$ *if and only if principal ideals are eventually idempotent (that is, for each $x \in R$ there is an integer n, depending on x, such that $Rx^n = Rx^{n+1}$).*

PROOF. If $\dim(R)=0$, then for each x there is a y such that $x - xyx$ is nilpotent. Expanding $x^n(1-yx)^n=0$, we get $x^n \in Rx^{n+1}$. Conversely, if principal ideals of R are eventually idempotent, the same holds for every homomorphic image of R. Clearly, then, R/P is a field for each prime P.

An interesting consequence of Proposition 4.2 is that taking direct products can increase Krull dimension. For example, the ideal generated by 2 in the ring $\prod_n \mathbf{Z}/(2^n)$ is not eventually idempotent.

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