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A REMARK ON THE COMMUTATIVITY OF CERTAIN RINGS

RAM AWTAR[1]

ABSTRACT. In a recent paper [1] Gupta proved that a division ring satisfying the polynomial identity $xy^2x=yx^2y$ is commutative. In this note our goal is to prove the following: If R is a semiprime ring with xy^2x-yx^2y central in R, for all x, y in R, then R is commutative.

Throughout this paper a ring will mean an associative ring. A result of Gupta [1] asserts that a division ring satisfying the polynomial identity $xy^2x=yx^2y$ is commutative. Our present object is to generalize Gupta's result as follows: If R is a semiprime ring in which xy^2x-yx^2y is in Z (center of R) for every x and y in R, then R is commutative. The author's proof does not depend on any well-known theorem, however, and so provides a different and elementary proof of the above result.

We begin with the following:

LEMMA. *Let R be a prime ring with xy^2x-yx^2y in Z for every x and y in R. Then R is commutative.*

PROOF. First we shall prove that $Z\neq(0)$. So we assume that $Z=(0)$. Then we have

(1) $$xy^2x = yx^2y, \quad \text{for every } x \text{ and } y \text{ in } R.$$

Replacing y by $y+y^2$ we obtain

(2) $$2xy^3x = y^2x^2y + yx^2y^2.$$

Since $y^2x^2y=y\cdot yx^2y=y\cdot xy^2x$, we get

(3) $$2xy^3x = yx(y^2x + xy^2).$$

If the characteristic is 2, then (3) becomes $yx(y^2x+xy^2)=0$. With $x=x+y$ this gives

(4) $$y^2(y^2x + xy^2) = 0.$$

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Then with $x=rx$ we get

(5) $$y^2(y^2rx + rxy^2) = 0.$$

Since $y^2 \cdot y^2r=y^2 \cdot ry^2$ from (4), (5) becomes $y^2[r(y^2x+xy^2)]=0$. We write this as $y^2R(y^2x+xy^2)=0$. Since R is prime, either $y^2=0$ or $y^2x+xy^2=0$, i.e. $y^2 \in Z=(0)$. Thus in either case $y^2=0$ for every y in R.

If the characteristic is not 2, we replace y by $y+y^3$ in (1) and get $2xy^4x=y^3x^2y+yx^2y^3$ or $2y^2x^2y^2=y^2xy^2x+xy^2xy^2$ (applying (1) to each term).

We write this as:

$$y^2x^2y^2 - y^2xy^2x = xy^2xy^2 - y^2x^2y^2$$

or

$$y^2x(xy^2 - y^2x) = (xy^2 - y^2x)xy^2.$$

We replace x by $x+y$ to get:

(6) $$y^3(xy^2 - y^2x) = (xy^2 - y^2x)y^3, \quad \text{for every } x \text{ and } y \text{ in } R.$$

Let I_{y^2} be the inner derivation by y^2, i.e. $x \mapsto xy^2-y^2x$, and I_{y^3} be the inner derivation by y^3. Then (6) becomes $I_{y^3}I_{y^2}(x)=0$. Thus the product of these derivations is again a derivation (the trivial one). Then by Theorem 1 in [3], we can conclude that either y^2 or y^3 is in Z, i.e. $=0$. If it is $y^3=0$, then (2) becomes: $y^2x^2y+yx^2y^2=0$. We set $x=x+y$ to get $2y^2xy^2=0$ or $y^2xy^2=0$ or $y^2Ry^2=0$. Then $y^2=0$.

Thus if $Z=0$ then $y^2=0$ for every y in R. Then $0=(x+y)^2x=xyx$, or $xRx=0$. Then $x=0$, $R=0$, a contradiction. Therefore $Z\neq(0)$.

Take $\lambda\neq 0$ in Z and let $x=x+\lambda$ in xy^2x-yx^2y in Z. We get $\lambda(xy^2-2yxy+y^2x)$ in Z. Since R is prime, we must then have

(7) $$xy^2 - 2yxy + y^2x \quad \text{in } Z,$$

for, if λa is in Z, then $\lambda ab-b\lambda a=0=\lambda(ab-ba)$. Then $R \cdot \lambda(ab-ba)=0=\lambda R(ab-ba)$, and since $\lambda\neq 0$, we have $ab-ba=0$, i.e. a is in Z.

In (7) we let $x=xy$ and get $(xy^2-2yxy+y^2x)y$ in Z. Then y is in Z unless $xy^2-2yxy+y^2x=0$. So if y is not in Z, $xy^2-2yxy+y^2x=0$ for every x in R, and if y is in Z then $xy^2-2yxy+y^2x$ is still 0. Therefore

(8) $$xy^2 + y^2x = 2yxy \quad \text{for every } x \text{ and } y \text{ in } R.$$

If the characteristic is $\neq 2$, then by the sublemma [**2**, p. 5] R is commutative.

If the characteristic is 2, then (8) becomes $xy^2+y^2x=0$ or y^2 is in Z for every y in R. Then $(x+y)^2=x^2+y^2+xy+yx$ is in Z or $xy+yx$ is in Z. Let $x=xy$ and get $(xy+yx)y$ is in Z. Then y is in Z unless $xy+yx=0$, which also means y is in Z. Thus $Z=R$ and R is commutative.

Let R be a semiprime ring in which xy^2x-yx^2y is in Z for every x, y in R. Since R is semiprime it is isomorphic to a subdirect sum of prime rings R_α each of which, as a homomorphic image of R, satisfies the hypothesis placed on R. By the above Lemma the R_α are commutative, hence R is commutative. Thus, we have proved

THEOREM. *Let R be a semiprime ring in which xy^2x-yx^2y is in Z, the center of R, for every x, y in R. Then R is commutative.*

Indeed, rings of 3×3 strictly upper triangular matrices over any ring satisfy the condition of the above theorem but these rings may not be commutative.

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DEPARTMENT OF MATHEMATICS, ALIGARH MUSLIM UNIVERSITY, ALIGARH (U.P.), INDIA