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FINITELY GENERATED STEADY $\mathfrak{N}$-SEMIGROUPS

TAKAYUKI TAMURA

ABSTRACT. In this paper the author proves that S is a finitely generated steady $\mathfrak{N}$-semigroup if and only if S is isomorphic to the direct product of a finite abelian group and the infinite cyclic semigroup; and also studies the homomorphisms of a finitely generated steady $\mathfrak{N}$-semigroup into another.

1. **Introduction.** By an $\mathfrak{N}$-semigroup we mean a commutative archimedean cancellative semigroup without idempotent. Following Petrich [9] an $\mathfrak{N}$-semigroup S is called steady if S cannot be embedded into another $\mathfrak{N}$-semigroup as a proper ideal. In this note the author determines the structure of finitely generated steady $\mathfrak{N}$-semigroups. Such a semigroup is isomorphic to the direct product of a finite abelian group and the infinite cyclic semigroup.

2. **Preliminaries.** Let P denote the set of all positive integers and P^0 the set of all nonnegative integers. The structure of $\mathfrak{N}$-semigroups was given by the author:

THEOREM 1 ([3], [11]). *Let G be an abelian group and $I: G\times G\to P^0$ be a function satisfying*

(1.1) $I(\alpha, \beta)=I(\beta, \alpha)$ *for all* $\alpha, \beta \in G$.

(1.2) $I(\alpha, \beta)+I(\alpha\beta, \gamma)=I(\alpha, \beta\gamma)+I(\beta, \gamma)$ *for all* $\alpha, \beta, \gamma \in G$.

(1.3) $I(\varepsilon, \alpha)=1$ (ε *being the identity of* G) *for all* $\alpha \in G$.

(1.4) *For each* $\alpha \in G$ *there is* $m \in P$ *such that* $I(\alpha, \alpha^m)>0$.

Let S be the set of all ordered pairs $\{x, \alpha\}$, $x \in P^0$, $\alpha \in G$. Define an operation in S by

$$\{x, \alpha\}\{y, \beta\} = \{x + y + I(\alpha, \beta), \alpha\beta\}.$$

Then S is an $\mathfrak{N}$-semigroup, denoted by $S=(G; I)$. Every $\mathfrak{N}$-semigroup can be obtained in this manner.

Let D be an $\mathfrak{N}$-semigroup and let $a \in D$. Define a relation ρ on D by $x \rho y$ if and only if $a^m x=a^n y$ for some $m, n \in P$. Then ρ is a congruence and $G=D/\rho$ is an abelian group and there exists $I: G\times G\to P^0$ such that

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$D \cong (G; I)$. G is called the structure group of D with respect to a. Thus G and I depend on an element a; so we denote these by G_a and I_a respectively if it is necessary to specify a. An $\mathfrak{N}$-semigroup S is called power joined if for every $a, b \in S$ there are $m, n \in P$ such that $a^m = b^n$.

PROPOSITION 2. (2.1) ([**1**], [**2**]). *An $\mathfrak{N}$-semigroup $S=(G;I)$ is power joined if and only if G is periodic.*

(2.2) ([**1**], [**2**], [**6**]). *$S=(G;I)$ is finitely generated if and only if G is finite.*

Let $S=(G;I)$ be a power joined $\mathfrak{N}$-semigroup. Let R denote the set of positive rational numbers. Define a function $\varphi: G \to R$ by

$$\varphi(\alpha) = \frac{1}{n}\sum_{i=1}^{n} I(\alpha, \alpha^i) \tag{2.3}$$

where n is the order of an element α of G.

PROPOSITION 3 [**10**]. *The function φ satisfies the following conditions*:

(3.1) $\varphi(\varepsilon)=1$, *ε the identity of G.*

(3.2) $\varphi(\alpha)+\varphi(\beta)-\varphi(\alpha\beta)$ *is a nonnegative integer, and*

(3.3) $I(\alpha, \beta)=\varphi(\alpha)+\varphi(\beta)-\varphi(\alpha\beta)$.

If S is finitely generated, equivalently, G is finite, then

$$\varphi(\alpha) = \frac{1}{|G|}\sum_{\xi \in G} I(\alpha, \xi) \quad \textit{where } |G| \textit{ is the order of } G. \tag{3.4}$$

There is a one-to-one correspondence between I and φ for a fixed G if G is periodic. Thus S is determined by G and φ, and it is denoted by $S=(G;\varphi)$. The notation "$S=(G;I)=(G;\varphi)$" means that φ corresponds to I. Let a be an element of S. The function φ corresponding to I_a is denoted by φ_a.

If G is finite, we can choose an element a of S such that

$$|G_a| \leqq \sum_{\xi \in G_a} I_a(\alpha, \xi) \quad \text{for all } \alpha \in G_a. \tag{3.5}$$

(See [7].) Then $(G_a; I_a)$ or $(G_a; \varphi_a)$ is called a *canonical* representation of S. Speaking of φ, $(G;\varphi)$ is canonical if and only if

$$\varphi(\alpha) \geqq 1 \quad \text{for all } \alpha \in G. \tag{3.6}$$

Let $S=(G;I)$ and $\Lambda(S)$ be the semigroup of all translations of S. Then $\Lambda(S)$ is a commutative cancellative semigroup. Let $\Gamma(S)$ be the subsemigroup of $\Lambda(S)$ consisting of all inner translations of S, and $\Psi(S)$ the archimedean component of $\Lambda(S)$ containing $\Gamma(S)$. Note that $S \cong \Gamma(S)$, that $\Psi(S)$ is also an $\mathfrak{N}$-semigroup, and that $\Psi(S)=\{\lambda \in \Lambda(S) : \lambda^n \in \Gamma(S)$ for some $n \in P\}$. Each element of $\Lambda(S)$ is determined by a pair $(m, \alpha) \in P^0 \times G$ where if $m=0$, we further require that $I(\alpha, \xi)>0$ for all $\xi \in G$.

The translation corresponding to (m, α) is denoted by $\lambda_{(m,\alpha)}$, and it takes an element $\{x, \xi\}$ of S to the element $\{x+m+I(\alpha, \xi)-1, \alpha\xi\}$ in S. The multiplication in $\Lambda(S)$ is given by

$$\lambda_{(m,\alpha)} \cdot \lambda_{(n,\beta)} = \lambda_{(m+n+I(\alpha,\beta)-1,\alpha\beta)}.$$

Then we can see that $\Gamma(S)=\{\lambda_{(m,\alpha)}: m>0, \alpha \in G\}$ and that $\Psi(S)=\Gamma(S)\cup A$ where $A=\{\lambda_{(0,\alpha)}: I(\alpha, \xi)>0$ for all $\xi \in G$ and $I(\alpha, \alpha^m)>1$ for some $m \in P\}$. See [**4**], [**5**] with respect to the translations of $\mathfrak{N}$-semigroups.

THEOREM 4 (PETRICH [**9**]). *The following conditions on an $\mathfrak{N}$-semigroup S are equivalent*:

(4.1) *For any $a, b \in S$, $aS\subseteq bS$ and $a^2S\subseteq b^2S$ imply $a \in bS$.*

(4.2) $\Psi(S)=\Gamma(S)$.

(4.3) *S cannot be embedded into an $\mathfrak{N}$-semigroup as a proper ideal.*

If an $\mathfrak{N}$-semigroup S satisfies one of (4.1), (4.2) and (4.3), S is called *steady*.

The condition (4.2) is equivalent to $A=\varnothing$. Hence we get the following lemma.

LEMMA 5. *An $\mathfrak{N}$-semigroup $S=(G; I)$ is steady if and only if $I(\alpha, \xi)>0$ for all $\xi \in G$ implies $I(\alpha, \alpha^m)=1$ for all $m \in P$.*

3. **Main result.** In this paper we treat only finitely generated $\mathfrak{N}$-semigroups.

THEOREM 6. *Let $S=(G; I)$ be a finitely generated $\mathfrak{N}$-semigroup. The following are equivalent*:

(6.1) *$I(\alpha, \xi)>0$ for all $\xi \in G$ implies $I(\alpha, \alpha^m)=1$ for all $m \in P$.*

(6.2) *$I(\alpha, \xi)>0$ for all $\xi \in G$ implies $\varphi(\alpha)=1$.*

(6.3) *$\varphi(\alpha)\leqq 1$ for all $\alpha \in G$.*

(6.4) *$I(\alpha, \xi)>0$ for all $\xi \in G$ implies $I(\alpha, \xi)=1$ for all $\xi \in G$.*

PROOF. (6.1)$\Rightarrow$(6.2): Obvious by (2.3).

(6.2)$\Rightarrow$(6.3): Let $\varphi(\beta)$ be the maximum of $\{\varphi(\xi): \xi \in G\}$. Suppose $I(\beta, \xi)=\varphi(\beta)+\varphi(\xi)-\varphi(\beta\xi)=0$ for some $\xi \in G$. Then $\varphi(\beta)<\varphi(\beta\xi)$. This is in contradiction to maximality of $\varphi(\beta)$. Accordingly $I(\beta, \xi)>0$ for all $\xi \in G$. By (6.2), $\varphi(\beta)=1$, hence $\varphi(\alpha)\leqq 1$ for all $\alpha \in G$.

(6.3)$\Rightarrow$(6.4): Assume $I(\alpha, \xi)>0$ for all $\xi \in G$. By (6.3) we have $0< I(\alpha, \xi)=\varphi(\alpha)+\varphi(\xi)-\varphi(\alpha\xi)\leqq 2-\varphi(\alpha\xi)<2$. Therefore $I(\alpha, \xi)=1$.

(6.4)$\Rightarrow$(6.1): Obvious.

As far as finitely generated $\mathfrak{N}$-semigroups are concerned, each of (6.1) through (6.4) is a necessary and sufficient condition for $S=(G; I)$ to be steady because of Lemma 5. Although the functions I and φ depend on the choice of standard elements, the condition in Lemma 5, (6.1), (6.2), (6.3) and (6.4) are the properties of S itself, independent of standard elements. The following theorem more explicitly describes the structure of S. Let $P\times G$ denote the direct product of the additive semigroup P and G. An element of $P\times G$ is denoted by (z, α), $z\in P$, $\alpha\in G$.

THEOREM 7. *A finitely generated $\mathfrak{N}$-semigroup S is steady if and only if S is isomorphic to the direct product of the positive integer semigroup P and a finite abelian group.*

PROOF. Assume that S is steady. Let $S=(G; I)=(G; \varphi)$ be a canonical representation of S. By (3.6) and (6.3), $\varphi(\alpha)=1$ for all $\alpha\in G$. Then $I(\alpha, \beta)=\varphi(\alpha)+\varphi(\beta)-\varphi(\alpha\beta)=1$ for all $\alpha, \beta\in G$. With respect to the representation of $S=(G; I)$,

$$\{x, \alpha\}\{y, \beta\} = \{x+y+1, \alpha\beta\}.$$

Define a map $f: S\to P\times G$ by $\{x, \alpha\}\mapsto(x+1, \alpha)$. We can easily see that f is an isomorphism.

Conversely let $S=P\times G$ where G is a finite abelian group. Let $a=(1, \varepsilon)$, ε being the identity of G. Then all elements prime to a have the form $(1, \alpha)$, $\alpha\in G$.

$$(1, \alpha)(1, \beta) = (1, \varepsilon)(1, \alpha\beta).$$

From this it follows that the structure group G_a is isomorphic to G and $I_a(\alpha, \beta)=1$ for all $\alpha, \beta\in G_a$. By Theorem 6 and Lemma 5 we conclude that S is steady.

REMARK. McAlister and O'Carroll characterized $P\times G$ in the more general case, that is, they proved in [**8**] that a finitely generated cancellative semigroup S without idempotent is isomorphic to $P\times G$ if and only if $S^2=Sa$ for all $a\in S\setminus S^2$.

REMARK. The equivalence of (6.1), (6.2) and (6.4) is valid even if $S=(G; I)$ is power joined.

4. **Homomorphisms.** Let $S=P\times G$ where G is a finite abelian group. It is easy to see that the structure group $G_{(m,\alpha)}$ of S with respect to an element (m, α) of S has order $m\cdot|G|$ and $G_{(1,\alpha)}$ has the smallest order of the structure groups of S. Furthermore $G_{(1,\alpha)}\cong G_{(1,\varepsilon)}\cong G$. Let G and H be finite abelian groups. It follows that $P\times G\cong P\times H$ if and only if $G\cong H$.

More generally we consider homomorphisms of one steady $\mathfrak{N}$-semigroup into another. Let $S=P\times G$, $T=P\times H$. Let (m, α) and $[x, \xi]$ denote elements of S and T respectively. Assume that f is a homomorphism of S into T. For each $\alpha\in G$, let $f(1, \alpha)=[p(\alpha), q(\alpha)]$ where $p: G\to P$, $q: G\to H$. Then

$$\begin{aligned} f(m, \alpha) &= f((1, \varepsilon)^{m-1}(1, \alpha)) = (f(1, \varepsilon))^{m-1}f(1, \alpha) \\ &= [(m-1)p(\varepsilon)+p(\alpha), q(\varepsilon)^{m-1}q(\alpha)] \end{aligned}$$

where ε is the identity of G. Likewise

$$f(n, \beta) = [(n-1)p(\varepsilon) + p(\beta), q(\varepsilon)^{n-1}q(\beta)], \quad \text{and}$$
$$f((m, \alpha)(n, \beta)) = [(m+n-1)p(\varepsilon) + p(\alpha\beta), q(\varepsilon)^{m+n-1}q(\alpha\beta)].$$

From $f((m, \alpha)(n, \beta)) = f(m, \alpha)f(n, \beta)$, we get

$$p(\alpha) + p(\beta) = p(\alpha\beta) + p(\varepsilon), \qquad q(\alpha)q(\beta) = q(\alpha\beta)q(\varepsilon) \quad \text{for all } \alpha, \beta \in G.$$

Let $r(\alpha) = p(\alpha) - p(\varepsilon)$ and $s(\alpha) = q(\alpha)q(\varepsilon)^{-1}$. Then we have $r(\alpha) + r(\beta) = r(\alpha\beta)$, $s(\alpha)s(\beta) = s(\alpha\beta)$, that is, r is a homomorphism of G into Z where Z is the group of integers under addition and s is a homomorphism of G into H. However, since G is finite, $r(\alpha) = 0$ for all $\alpha \in G$, hence $p(\alpha) = p(\varepsilon)$ for all $\alpha \in G$. Let $t = p(\varepsilon)$, $\sigma = q(\varepsilon)$. We get $f(m, \alpha) = [tm, \sigma^m s(\alpha)]$.

Conversely let s be a homomorphism of G into H, σ a fixed element of H and t a fixed positive integer. Define $f: S \to T$ by

$$f(m, \alpha) = [tm, \sigma^m s(\alpha)]. \tag{8}$$

Then it is easy to show that f is a homomorphism. Thus all homomorphisms of S into T are determined by $t \in P$, $\sigma \in H$, and $s \in \mathrm{Hom}(G, H)$. Moreover we see from (8) that f is one-to-one if and only if s is one-to-one; f is onto if and only if $t = 1$ and s is onto. Let $\mathrm{Hom}(S, T)$ denote the semigroup of all homomorphisms of S into T in the usual sense. Clearly $\mathrm{Hom}(S, T) \neq \varnothing$. Consequently we have the following theorem.

THEOREM 9. (9.1) $\mathrm{Hom}(P \times G, P \times H) \cong P \times H \times \mathrm{Hom}(G, H)$ *hence it is a finitely generated steady* $\mathfrak{N}$*-semigroup.*

(9.2) $P \times H$ *is a homomorphic image of* $P \times G$ *if and only if* G *is homomorphic onto* H.

(9.3) $P \times G$ *is isomorphic into* (*onto*) $P \times H$ *if and only if* G *is isomorphic into* (*onto*) H.

Let $|G| < \infty$ and $S = (G; I)$. According to **[6]**, S is isomorphic to a subdirect product of a positive integer additive semigroup and G, hence S can be embedded into $P \times G$ in the natural way. Consider the category of the embeddings of S into finitely generated $\mathfrak{N}$-semigroups. Even if $S = (G; I)$ is canonical, the embedding $S \to P \times G$ need not be a universal object. (See the example below.) We state the following theorem without detailed proof.

THEOREM 10. *Let* $S = (G; I) = (G; \varphi)$, $|G| < \infty$, *and let* $G_0 = \{\alpha \in G: \varphi(\alpha) \in P\}$. *Then* G_0 *is a subgroup of* G *and* S *is isomorphic to a subsemigroup of* $P \times G_0$ *and the embedding* $S \to P \times G_0$ *is a universal repelling object in the category of the embeddings of* S *into finitely generated steady* $\mathfrak{N}$*-semigroups.*

$Z\times G_0$ is isomorphic to the quotient group of S, and G_0 is isomorphic to the torsion subgroup of $Z\times G_0$. It is easy to see that $S\to P\times G_0$ is a universal object, but G_0 need not be a structure group of S. (See the example below.) Related to Theorem 10, see [8] and [12].

EXAMPLE. Let G be the Klein four group: $\alpha^2=\beta^2=\varepsilon$. Define φ by

$$\varphi(\varepsilon) = \varphi(\alpha)=1, \qquad \varphi(\beta) = \tfrac{3}{2}, \qquad \varphi(\alpha\beta) = \tfrac{5}{2}.$$

Let $S=(G;\varphi)$. S has two canonical representations with respect to $\{0, \varepsilon\}$ and $\{0, \alpha\}$. Then $G_{\{0,\varepsilon\}}\cong G$ but $G_{\{0,\alpha\}}$ is a cyclic group of order 4, hence $P\times G\not\cong P\times G_{\{0,\alpha\}}$. Since $|G_{\{0,\beta\}}|=6$, $P\times G$ cannot be embedded into $P\times G_{\{0,\beta\}}$ but S can be embedded into $P\times G_{\{0,\beta\}}$. G_0 consists of ε and α, and G_0 is not a structure group of S.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS, CALIFORNIA 95616