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ON CONTINUITY OF INVARIANT MEASURES

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ABSTRACT. MAIN THEOREM. *Let Φ be a set of transformations on a set X. The following conditions are then equivalent*:

(1) *There is a noncontinuous finitely additive measure defined on all subsets of X and invariant under all transformations in Φ.*

(2) *There is an integer m such that for any finite subset F of Φ there is a finite subset A_F of X, with no more than m elements, such that each f in F acts as a permutation on A_F.*

1. **Introduction.** Let X be a nonempty set, and let $\mathscr{A}$ be an algebra of subsets of X. By a measure μ on $(X, \mathscr{A})$ we mean a finitely additive nonnegative set function on $\mathscr{A}$ such that $\mu(X)=1$. Let Φ be a family of functions from X to X such that for any f in Φ and any U in $\mathscr{A}$, $f^{-1}U$ is in $\mathscr{A}$ (that is, the algebra $\mathscr{A}$ is invariant under Φ). The measure μ is Φ-invariant if $\mu(f^{-1}U)=\mu(U)$ for any f in Φ and any U in $\mathscr{A}$. If V is in $\mathscr{A}$, the range of μ on V is the set of all numbers $\mu(U)$, where U ranges over all subsets of V in $\mathscr{A}$. In this paper we find some conditions on Φ, $\mathscr{A}$ that ensure that for any Φ-invariant measure μ on $\mathscr{A}$ and any V in $\mathscr{A}$, μ has full range $[0, \mu(V)]$ on V. A complete characterization is reached in the case that $\mathscr{A}$ is the algebra of all subsets of X. There are only partial results for the general case.

There are a number of papers that deal with special cases of our problem. They look at the situation in which X is a topological semigroup, $\mathscr{A}$ is the algebra of Borel subsets of X, and Φ is X (acting on itself by left multiplication). Granirer in [4] obtained an almost complete answer for discrete semigroups. A remaining problem was settled by Chou in [3]. Recently, Snell has in [6] extended the results of Granirer and Chou to locally compact groups. By looking at the problem in a more general setting, we are able to extend these various results and, to some degree, to simplify their proofs.

2. **Some preliminary results about Φ-invariant measures.** The measure μ is said to be continuous if for every $\varepsilon>0$, there exists a finite partition $X=X_1\cup\cdots\cup X_n$ of X, with the X_i in $\mathscr{A}$, such that $\mu(X_i)<\varepsilon$ for all i.

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LEMMA 1 (SOBCZYK AND HAMMER [7]). *Let μ be a continuous measure on $(X, \mathscr{A})$. Then there is a dense subset S of* $[0, 1]$, *and a collection* $\{U_s\}$ *of elements of $\mathscr{A}$, where s ranges over S such that* (i) *if $s<t$, $U_s \subset U_t$, and* (ii) $\mu(U_s)=s$ *for all s in S. If $\mathscr{A}$ is a σ-algebra, S may be taken to be all of* $[0, 1]$.

So if μ is continuous, and $\mathscr{A}$ is a σ-algebra, then μ attains full range, and moreover does so on a nested collection of sets.

LEMMA 2 (SOBCZYK AND HAMMER [7]). *Any measure μ has a unique decomposition (apart from the order of the terms) in the form $\mu=\kappa+\sum_{i=1}^{\infty} a_i\lambda_i$, where κ is continuous, the a_i are nonnegative, and the λ_i are* $\{0, 1\}$*-valued finitely independent measures.*

For any measure μ, and any $f:X\rightarrow X$ under which $\mathscr{A}$ is invariant, define $f\mu$ by $(f\mu)(U)=\mu(f^{-1}U)$. Then $f\mu$ is a measure on $(X, \mathscr{A})$. If μ is $\{0, 1\}$-valued, so is $f\mu$. If $\mu=\sum_{i=1}^{\infty} a_i\mu_i$, then $f\mu=\sum_{i=1}^{\infty} a_i(f\mu_i)$. Verification of these facts is straightforward.

Let $\mu=\kappa+\sum_{i=1}^{\infty} a_i\lambda_i$ be the Sobczyk-Hammer decomposition of μ, where we choose $a_i \geqq a_{i+1}$ for all i. If μ is f-invariant, then $\mu=f\mu=f\kappa+\sum_{i=1}^{\infty} a_i(f\lambda_i)$. Let $\lambda=\sum_{i=1}^{\infty} a_i\lambda_i$. Now $f\lambda$ is a linear combination of $\{0, 1\}$-valued measures, and $(f\lambda)(X)=\lambda(X)$. By the uniqueness of the Sobczyk-Hammer decomposition, we have $f\lambda=\lambda$, and so $f\kappa=\kappa$. If μ is not continuous, $\mu\neq\kappa$, and so not all the a_i are zero. Let n be the integer such that $a_1=a_2=\cdots=a_n$ but $a_{n+1}<a_n$ (recall the a_i are nonincreasing). Again from the uniqueness of the decomposition, we see that $f\lambda_i=\lambda\pi(i)$ for $1\leqq i\leqq n$, where π is some permutation of $\{1, \cdots, n\}$. Hence the measure $n^{-1}(\lambda_1+\cdots+\lambda_n)$ is f-invariant. The argument is independent of the particular f selected. So we have proved:

LEMMA 3. *If $(X, \mathscr{A})$ admits a noncontinuous Φ-invariant measure, then $(X, \mathscr{A})$ admits a Φ-invariant measure of form $1/n(\lambda_1+\cdots+\lambda_n)$, where the λ_i are* $\{0, 1\}$*-valued independent measures.*

Lemma 3 gives a substantial reduction of the original problem. It yields, for instance, a simplified proof of the following recent result:

THEOREM (SNELL [6]). *Let S be an infinite subsemigroup of a locally compact topological group. Let $\mathscr{A}$ be the algebra of Borel sets of S, and let Φ be S acting on itself by left multiplication. Then every Φ-invariant measure on $(S, \mathscr{A})$ attains range* $[0, 1]$ *on a nested collection of sets.*

PROOF. By Lemma 1 and Lemma 3, it is sufficient to show that $(S, \mathscr{A})$ does not admit a Φ-invariant measure of form $n^{-1}(\lambda_1+\cdots+\lambda_n)$, where the λ_i are $\{0, 1\}$-valued. But this follows at once from Theorem 3 of [5]. The argument actually shows every Baire measure is already continuous.

The next lemma is a localization result that enables one to go from infinite sets of functions to finite sets.

LEMMA 4. *If* $(X, \mathscr{A})$ *admits an F-invariant measure* μ_F *for every finite subset F of* Φ, *then* $(X, \mathscr{A})$ *admits a* Φ*-invariant measure* μ. *Moreover, if each* μ_F *takes on no more than* $m+1$ *values on subsets of X*, μ *can be chosen with the same property.*

PROOF. A technique from nonstandard analysis is used to take the appropriate limit. More detail about the ideas involved can be found for instance in [**1**]. Let I be the collection of finite subsets of Φ. For any f in Φ, let K_f be the set of F in I such that f is in F. The family $\{K_f\}$ has the finite intersection property. Let D be an ultrafilter on I that extends this family. For any U in $\mathscr{A}$, define $\mu_U: I \rightarrow R$ by $\mu_U(F)=\mu_F(U)$. Define a measure μ on $(X, \mathscr{A})$ by $\mu(U)=st(\mu_U/D)$, where st is the ordinary standard part function of nonstandard analysis. For any U and any F, $0 \leqq \mu_U(F) \leqq 1$, so the standard part of μ_U/D exists. It is easy to verify μ is a measure. For any f in Φ, $\{F: \mu_F(U)=\mu_F(f^{-1}U)\} \supseteq K_f$, and hence lies in D. Therefore $\mu_U/D=\mu_{f^{-1}U}/D$, and so μ is Φ-invariant. Since D is an ultrafilter, for any partition of I into $m+1$ sets, one of the sets must lie in D. So if each μ_F takes on no more than $m+1$ values, the same is true of μ.

3. **Analysis of the discrete case.** In this section, $\mathscr{A}$ will be the algebra of all subsets of X. We characterize those Φ for which there exists a Φ-invariant discontinuous measure. The main additional tool is the following:

LEMMA 5. *Let* λ *be a* $\{0, 1\}$*-valued measure on the collection of all subsets of X, and let* $f: X \rightarrow X$ *be such that* $f\lambda=\lambda$. *Then f is the identity map on a set of* λ*-measure* 1.

PROOF. The result has been known for a fairly long time, having been proved by (among others) Keisler and M. Rudin. A proof appears in print in [**2**] for countable X. That proof works equally well for arbitrary X.

THEOREM. *The following are equivalent conditions on* Φ:

(1) *X supports a* Φ*-invariant noncontinuous measure.*

(2) *There is an integer m such that for any finite subset F of* Φ *there is a subset* A_F *of X, with no more than m elements, such that each f in F acts as a permutation on* A_F.

PROOF. We first show that if (2) holds, so does (1). Define measure μ_F on X by assigning equal nonzero mass to the objects of A_F and normalizing so that $\mu_F(A_F)=1$. Since each f in F permutes A_F, μ_F is F-invariant on X, and takes on no more than $m+1$ values. By Lemma 4, there is then a Φ-invariant measure on X taking on no more than $m+1$ values.

Next we show that if (1) holds, so does (2). If (1) holds, then by Lemma 3 X supports a Φ-invariant measure of form $\mu=n^{-1}(\lambda_1+\cdots+\lambda_n)$, where the λ_i are $\{0, 1\}$-valued. If μ is f-invariant and g-invariant, it is invariant under the composition $f\circ g$ of f and g. So we may assume Φ is a semigroup. We may further assume that Φ contains the identity map e.

Each f in Φ induces a permutation on the λ_i. Put $f\sim g$ if $f\lambda_i=g\lambda_i$ for $1\leqq i\leqq n$. The relation $\sim$ is a congruence on the semigroup Φ. The equivalence classes under the natural multiplication form a group G isomorphic to a subgroup of the permutation group on n letters. Let m be the order of G.

We need to characterize the relation $\sim$ more closely. If f and g agree on a set of μ-measure 1, then clearly $f\sim g$. The converse also holds. If $f\sim e$, then $f\lambda_i=e\lambda_i=\lambda_i$ for $1\leqq i\leqq n$. Hence by Lemma 5, f is the identity on a set of λ_i-measure 1 for all i, and so f is the identity on a set of μ-measure 1. In general, if $f\sim g$, then since G is a group, $hf\sim hg\sim e$ for some h. But then h must be one-to-one on a set of μ-measure 1. It follows that $f=g$ on a set of μ-measure 1.

Now let F be a finite subset of Φ. The relation $\sim$ partitions F into sets $F_1, \cdots, F_m$. It is convenient here to assume that each element of G is represented in F. There exists then a set U of μ-measure 1 such that

(i) If f, g are in F_i, then f and g agree on U.

(ii) If f is in F_i and g is in F_j, where $i\neq j$, then f and g differ at every point of U.

This follows from our characterization of the relation $\sim$ and from the fact that the intersection of a finite number of sets of measure 1 has measure 1.

Write down now all true relations of the form $f\circ g\sim h$, where f, g, h range over F. There is only a finite number of these. Therefore there is a set V of μ-measure 1 such that for any x in V, and any relation $f\circ g\sim h$, $(f\circ g)x=h(x)$.

So $U\cap V$ has μ-measure 1, and so there is a point x in it. Let A_F be the set of points of form fx, where f is in F. By (i) and (ii), A_F has precisely m elements. It remains to show that each element of F permutes A_F.

We show that if f is in F, and y is in A_F, then fy is in A_F. Now $y=gx$ for some g in F, and so $fy=(f\circ g)x$. But since F contains a representative for each element of G, $f\circ g\sim h$ for some h in F. From our choice for V, $(f\circ g)x=hx$, which is in A_F. If y and z are in A_F and $y\neq z$, $fy\neq fz$. For let $y=gx$, $z=hx$. f and g cannot be equivalent. So $f\circ g$ and $f\circ h$ are not equivalent, and so $(f\circ g)x\neq(f\circ h)x$. This completes the proof.

4. Some remarks about the general case. Let X be a compact Hausdorff space (or more generally an H-closed space). Let $\mathscr{A}$ be the algebra generated by the open sets, and let Φ be a family of continuous functions on X. Suppose $(X, \mathscr{A})$ admits a discontinuous Φ-invariant measure. By Lemma 3, $(X, \mathscr{A})$ admits a Φ-invariant measure of form

$$\mu = n^{-1}(\lambda_1 + \cdots + \lambda_n),$$

where the λ_i are $\{0, 1\}$-valued measures (ultrafilters) on it. Let p_i $(1 \leqq i \leqq n)$ be the limit points of the λ_i. Each f in Φ permutes the λ_i, and so since f is continuous it permutes the p_i. Conversely, if there is a finite set P such that each f in F permutes P, $(X, \mathscr{A})$ admits a discontinuous Φ-invariant measure. This result is not nearly as informative as the results for the discrete case. The ideas used in §3 can be used to give a construction of all Φ-invariant measures that take on a finite number of values. We cannot give such a general construction in compact spaces.

The main obstruction to deeper knowledge is our failure to prove an analogue of Lemma 5 in a more general setting. Lemma 5 fails in some compact Hausdorff spaces (the one point compactification of the discrete space of cardinality ω_1 provides an example). We conjecture Lemma 5 holds for locally compact metric spaces and continuous maps. We have verified it does hold in a number of cases, the most interesting one being the real line and continuous monotone maps.

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