PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 41, Number 2, December 1973

AN EXAMPLE RELATING TO ARHANGEL'SKIĬ'S CLASS MOBI

JAMES R. BOONE

ABSTRACT. A space in Arhangel'skiĭ's class MOBI is presented which fails to have two structural properties which are weaker than countable metacompactness. A space is countably metacompact if and only if it is countably θ-refinable and pointwise collectionwise normal for countable collections. This space does not have either of these properties.

1. **Introduction.** Arhangel'skiĭ introduced the class MOBI and posed a series of questions about the properties of this class of topological spaces in [**1**]. Bennett provided a characterization of the class MOBI and answered most of these questions negatively in [**2**].

In this paper, a space in MOBI is studied, in §3, which is not countably θ-refineable and not pointwise collectionwise normal for countable collections. In §2, countably metacompact spaces are characterized as those spaces which are both countably θ-refinable and pointwise collectionwise normal for countable collections.

The example we will consider is the space (F, ψ) defined by Wicke and Worrell [**9**, Example 3]. It is established in [**9**] that (F, ψ) is the open compact continuous image of a metacompact complete Moore space. Hanai [**7**, Theorem 5] has shown that every metacompact developable Hausdorff space is an open compact image of a metric space. Since the open compact image of an open compact image of a metric space is in MOBI, (F, ψ) is in MOBI. The space (F, ψ) has a point-countable base of countable order and has a λ-base [**9**].

The space (F, ψ) is weakly θ-refinable, but not θ-refinable [**4**, Example 1] and not countably subparacompact [**3**, Theorem 3.1]. Hodel [**8**, Theorem 3] has shown that every countably subparacompact space is countably metacompact. It is shown in this paper that (F, ψ) fails to have two properties that are weaker than countable metacompactness, which together imply countable metacompactness.

Received by the editors October 14, 1972.

AMS (MOS) subject classifications (1970). Primary 54G20; Secondary 54D20, 54D15.

Key words and phrases. Arhangel'skiĭ's class MOBI, θ-refinable, pointwise collectionwise normal, countably metacompact.

2. **Preliminaries to the example.** The notion of a θ-refinement was introduced by Wicke and Worrell in [**10**]. A space is defined as *countably θ-refinable* if every countable open covering of the space has a θ-refinement. Pointwise collectionwise normal spaces are defined in [**5**] and used to characterize metacompactness in the class of θ-refinable spaces.

THEOREM 2.1 [**5**]. *A space is metacompact if and only if it is θ-refinable and pointwise collectionwise normal.*

A space is said to be *pointwise collectionwise normal for countable collections* if for each discrete collection of closed sets $\mathscr{F}=\{F_i:i\in N\}$, there exists a point-finite collection of open sets $\mathscr{G}=\{G_i:i\in N\}$ such that $F_i\subset G_i$, for each $i\in N$ and $F_i\cap G_j=\varnothing$, if $i\neq j$. A *σ-precise refinement* of a covering $\mathscr{U}=\{U_\alpha:\alpha\in A\}$ is defined as a collection $\mathscr{V}=\bigcup_{n=1}^{\infty}\mathscr{V}_n$, where for each $n\in N$, $\mathscr{V}_n=\{V_\alpha(n):\alpha\in A\}$ and $V_\alpha(n)\subset U_\alpha$, for each $\alpha\in A$. The standard method of generating precise refinements [**6**, p. 162] yields that every open covering of a (weakly) θ-refinable space has an open σ-precise (weak) θ-refinement.

THEOREM 2.2. *A space is countably metacompact if and only if it is countably θ-refinable and pointwise collectionwise normal for countable collections.*

PROOF. Clearly, every countably metacompact space is countably θ-refineable and pointwise collectionwise normal for countable collections. To prove the converse, consider a σ-precise θ-refinement $\mathscr{C}$ of a countable open covering $\mathscr{H}$ of a countably θ-refinable space X. In the proof of Theorem 2.1, as presented in [**5**, Theorem 3.1], the discrete collections of closed sets $\mathscr{F}^{ij}$ are indexed by finite subsets γ, card$(\gamma)=j$, of the indexing set A_i for the layer $\mathscr{C}_i$ in the σ-precise θ-refinement $\mathscr{C}$. Since $\mathscr{H}$ is a countable open covering of X, each indexing set A_i, in the proof of Theorem 3.1 of [**5**] can be taken to be N. Hence, only countable discrete collections of closed sets are generated. Since X is pointwise collectionwise normal for countable collections, the construction of a point-finite open refinement proceeds exactly as in the proof of Theorem 2.1.

3. **Example.** *A space in* MOBI *which is neither countably θ-refinable nor pointwise collectionwise normal for countable collections.* Let X be the set of real numbers. For each irrational number u, let $\{\{u\}\}$ be the neighborhood base at u. For each rational number p, the collection of sets of the form $\{x\in X:x=p$ or x is irrational and $x\in(a,b)\}$ where $p\in(a,b)$, will be the neighborhood base at p. The space X, generated in this manner, is the space (F,ψ) of Wicke and Worrell [**9**, Example 3].

X is not pointwise collectionwise normal for the countable discrete collection of rational singletons. Let $\{p_i:i\in N\}$ be the set of rationals in X,

and let I be the set of irrationals in X. Then $\{\{p_i\}:i\in N\}$ is a countable discrete collection of closed subsets of X. For each $i\in N$, let U_i be any basic open neighborhood of p_i. Then $\{p_i\}\subset U_i$, for each $i\in N$, and $\{p_i\}\cap U_j=\varnothing$, if $i\neq j$. Let $\mathscr{U}=\{U_i:i\in N\}$, and for each $k=0, 1, 2, \cdots$, let $I_k=\{u\in I:u$ is an element of at most k sets in $\mathscr{U}\}$. Assume that for some n, I_n, as a subset of the reals with the usual topology, is such that $\operatorname{Int}(\operatorname{cl}(I_n))\neq\varnothing$. Let (a, b) be any open interval contained in $\operatorname{Int}(\operatorname{cl}(I_n))$, and let p_{i_1} be any rational in (a, b). Then $U_{i_1}\cap(a, b)$ is a basic open neighborhood of p_{i_1}, and $U_{i_1}=(I\cap(a_{i_1}, b_{i_1}))\cup\{p_{i_1}\}$, for some $a_{i_1}<b_{i_1}$. Let $r_1=\max\{a, a_{i_1}\}$ and $s_1=\min\{b, b_{i_1}\}$. Then $(r_1, s_1)\subset(a, b)$, and $(r_1, s_1)\cap I\subset U_{i_1}$. Suppose rational numbers p_{i_k} have been selected from (r_{k-1}, s_{k-1}), and $(r_k, s_k)\cap I\subset U_{i_k}$ for each k, $1<k\leqq n$, and $(r_n, s_n)\subset(r_{n-1}, s_{n-1})\subset\cdots\subset(r_1, s_1)\subset(a, b)$. Let $p_{i_{n+1}}$ be any rational in (r_n, s_n). Then $U_{i_{n+1}}\cap(r_n, s_n)$ is a basic open neighborhood of $p_{i_{n+1}}$, and $U_{i_{n+1}}=(I\cap(a_{i_{n+1}}, b_{i_{n+1}}))\cup\{p_{i_{n+1}}\}$, for some $a_{i_{n+1}}<b_{i_{n+1}}$. Let $r_{n+1}=\max\{r_n, a_{i_{n+1}}\}$ and $s_{n+1}=\min\{s_n, b_{i_{n+1}}\}$. Then $(r_{n+1}, s_{n+1})\subset(r_n, s_n)$ and $(r_{n+1}, s_{n+1})\cap I\subset U_{i_j}$, for each $j=1, 2, \cdots, n+1$. Any rational p in (r_{n+1}, s_{n+1}) is a cluster point of I_n, because $p\in(r_{n+1}, s_{n+1})\subset(a, b)\subset\operatorname{Int}(\operatorname{cl}(I_n))$. Thus, there is a point $u\in I_n$ such that $u\in(r_{n+1}, s_{n+1})$. Accordingly, $u\in U_{i_j}$, for each $j=1, 2, \cdots, n+1$. This is a contradiction to the definition of I_n. Hence, I_n, as a subset of the reals with the usual topology, is nowhere dense for each $n\in N$. Since I is a second category set $I\neq\bigcup_{n\in N} I_n$. Thus, there exists some irrational u, such that $u\notin I_n$, for each $n\in N$. Then u is a point in infinitely many sets in $\mathscr{U}$. Hence, X is not pointwise collectionwise normal, even for a countable discrete collection of singletons.

X is not countably θ-refinable. Consider any open covering of X, which covers each rational number with a basic open set. Consider any sequence of open coverings $\mathscr{G}_i$, $i\in N$, such that $\mathscr{G}_i$ is a refinement of the original covering, for each $i\in N$. The preceding argument establishes that the set $I_{n,k}=\{u\in I:u$ is an element of at most n sets in $\mathscr{G}_k\}$ is nowhere dense in X, as a subset of the reals with the usual topology. Again, since I is second category, there is some irrational u such that $u\notin I_{n,k}$, for each n and k in N. Thus, u is an element of infinitely many sets in each $\mathscr{G}_k$. Hence, no open covering, even a countable open covering, of X, which covers each rational with a basic open set, has a θ-refinement. Accordingly, X is not countably θ-refinable.

The author wishes to thank the referee of this paper for many helpful suggestions.

References

1. A. Arhangel'skiĭ, *Mappings and spaces*, Uspehi Mat. Nauk **21** (1966), no. 4 (130), 133–184=Russian Math. Surveys **21** (1966), no. 4, 115–162. MR **37** #3534.

2. H. R. Bennett, *On Arhangel'skiĭ's class* MOBI, Proc. Amer. Math. Soc. **26** (1970), 178–180. MR **42** #2425.

3. H. R. Bennett and E. S. Berney, *Subparacompactness and G_δ-diagonals in Arhangel'skiĭ's class* MOBI, Proc. Amer. Math. Soc. **30** (1971), 573–577. MR **44** #990.

4. H. R. Bennett and D. J. Lutzer, *A note on weak θ-refinability*, General Topology and Appl. **2** (1972), 49–54.

5. J. R. Boone, *A characterization of metacompactness in the class of θ-refinable spaces*, General Topology and Appl. **3** (1973).

6. J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1966. MR **33** #1824.

7. S. Hanai, *On open mappings*. II, Proc. Japan Acad. **37** (1961), 233–238. MR **23** #A4116.

8. R. E. Hodel, *A note on subparacompact spaces*, Proc. Amer. Math. Soc. **25** (1970), 842–845. MR **41** #7629.

9. H. H. Wicke and J. M. Worrell, Jr., *Open continuous mappings of spaces having bases of countable order*, Duke Math. J. **34** (1967), 255–271. MR **35** #979.

10. J. M. Worrell, Jr. and H. H. Wicke, *Characterizations of developable topological spaces*, Canad. J. Math. **17** (1965), 820–830. MR **32** #427.

Department of Mathematics, Texas A&M University, College Station, Texas 77843