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A NOTE ON STRASSEN'S VERSION OF THE LAW OF THE ITERATED LOGARITHM

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ABSTRACT. Strassen's law of the iterated logarithm is extended to stationary ergodic martingales and to a non-identically-distributed case.

1. **Introduction.** V. Strassen [6] proved that if $\{x_n\}$ are i.i.d.r.v.'s with $Ex=0$, $Ex^2=1$, then for a continuous map ϕ from $C[0, 1]$ to R^1, the sequence $\phi(\eta_n)$ is relatively compact and the set of its limit points coincides with $\phi(K)$ where η_n is obtained by linearly interpolating

$$(2n \operatorname{Log} \operatorname{Log} n)^{-1/2} S_i \quad \text{at } t = i/n,\ i = 1, 2, \cdots, n, \quad \text{and } S_i = x_1 + \cdots + x_i.$$

K is the set of absolutely continuous functions $f \in C[0, 1]$ such that $f(0)=0$ and $\int_0^1 |f'(t)|^2\, dt \leqq 1$.

The key idea of the paper is that Strassen's theorem holds for any random sequence for which there is a Skorohod embedding in Brownian motion with stopping times τ_n satisfying the strong law of large numbers. Two examples are given: one is the stationary ergodic martingales and the other a non-identically-distributed case.

As an easy extension of the corollary of Strassen [6], we can obtain in particular if $\{x_i, F_i, i \geqq 1\}$ is a stationary ergodic stochastic sequence $E(x_i | F_{i-1})=0$ a.s. for all $i \geqq 1$ and $Ex_1^2=1$, then

$$\limsup S_n (2n \log\log n)^{-1/2} = 1 \text{ a.s.} \tag{1}$$

and

$$\limsup \frac{1}{n} \sum_{i=1}^{n} S_i \Big/ \left(\tfrac{2}{3} n \log\log n\right)^{1/2} = 1 \text{ a.s.} \tag{2}$$

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(1) is recently obtained by W. F. Stout [7] by a different method and (2) is an extension of the iterated logarithm for "Cesàro sums" considered by Gal and Stackelberg [5], and Gaposhkin [2].

2. Results.

LEMMA 1. *If* $\{\tau_i\}$, *the sequence of stopping times of the Skorohod representation theorem, satisfies the strong law of large numbers namely*

$$(\tau_1 + \cdots + \tau_k)/k \to 1 \text{ a.s.},$$

then there is a probability space with a Brownian motion $B(t)$ *and a sequence* $\{\tilde{S}_n\}_1^\infty$ *having the same distribution as* $\{S_n\}_1^\infty$ *such that*

$$\sup_{\tau \leq t} \frac{|\tilde{S}_\tau - B(\tau)|}{(2t \log \log t)^{1/2}} \to 0 \quad \text{a.s. as } t \to \infty. \tag{3}$$

PROOF. Follows from [6].

THEOREM 1. *Strassen's theorem* [6] *holds for a stationary ergodic martingale difference sequence* $\{x_n, F_n, n \geq 1\}$ *with* $Ex_1^2 = 1$.

To facilitate the proof of Theorems 1 and 2 let us state some known results from Jonas's thesis [3] and often we shall sketch a proof.

A generalization of Skorohod's theorem. Let (A) $\hat{X}_1, \hat{X}_2, \cdots$ be a sequence of random variables on a probability space $(\hat{\Omega}, \hat{\mathscr{B}}, \hat{P})$ for which $E(\hat{X}_n^2 | \hat{X}_1, \cdots, \hat{X}_n)$ exists for all n and $E(\hat{X}_n | \hat{X}_1, \cdots, \hat{X}_n) = 0$ a.s.

Let $(\Omega, \mathscr{B}, P)$ be a probability space with the following properties:

(1) There exists a Brownian motion $B = B(t, \omega)$ on $(\Omega, \mathscr{B}, P)$.

(2) There exist random variables independent of the Brownian motion B, $Y_1, Y_2, \cdots$ on $(\Omega, \mathscr{B}, P)$ which are pairwise independent and distributed uniformly over the interval $[0, 1]$.

Then there exists a sequence of nonnegative random variables $\tau_1, \tau_2, \cdots$ on $(\Omega, \mathscr{B}, P)$ and (B) random variables $X_1, X_2, \cdots$ on $(\Omega, \mathscr{B}, P)$, such that r.v.'s of the sequence (A) and (B) have the same distribution and $\sum_1^n X_i = B(\sum_1^n \tau_i)$ a.s.

Moreover, if $\mathscr{A}_n = \mathscr{B}(X_1, X_2, \cdots, X_n, B(t), 0 \leq t \leq \sum_{j=1}^n \tau_j)$, i.e. the σ-field generated by $X_1, \cdots, X_n$ and $B(t)$ $(0 \leq t \leq \sum_1^n \tau_j)$, then the following hold.

(1) τ_n is $\mathscr{A}_n$-measurable.

(2) For each $s > 0$, $B_n(s) = B(\sum_{j=1}^n \tau_j + s) - B(\sum_{j=1}^n \tau_j)$ is independent of $\mathscr{A}_n$.

(3) $E(\tau_n | \mathscr{A}_{n-1})$ exists and $E(\tau_n | \mathscr{A}_{n-1}) = E(X_n^2 | \mathscr{A}_{n-1})$ a.s. $= E(X_n^2 | X_1, X_2, \cdots, X_{n-1})$.

(4) If $k>0$ and if $E(X_n^{2k}|X_1, \cdots, X_{n-1})$ exists, then $E(\tau_n^k|\mathscr{A}_{n-1})$ exists too, and further

$$E(\tau_n^k \mid \mathscr{A}_{n-1}) \leqq L_k E(X_n^{2k} \mid \mathscr{A}_{n-1}) = L_k E(X_n^{2k} \mid X_1, \cdots, X_{n-1}) \quad \text{a.s.}$$

where L_k is a constant depending only on k.

SKETCH OF PROOF. Like Skorohod [**4**, pp. 163–164], let us state

LEMMA 2. *Suppose that $B(t)$ is a Brownian motion process for $t \geqq 0$ and $B(0)=0$ and τ_{n+1} is the smallest root of $(B_n(t)-a)(B_n(t)-b)=0$, where $a<0<b$. Then for every $\lambda>0$,*

$$E(\exp(-\lambda\tau_{n+1})I_{[B_n(\tau_{n+1})=b]} \mid \mathscr{A}_n) = \sinh b(2\lambda)^{1/2}/\sinh_, b-a)(2\lambda)^{1/2} \quad a.s.$$

and

$$E(\exp(-\lambda\tau_{n+1})I_{[B_n(\tau_{n+1})=b]} \mid \mathscr{A}_n) = -\sinh a(2\lambda)^{1/2}/\sinh(b-a)(2\lambda)^{1/2} \quad a.s.$$

We can prove the lemma and the theorem by induction i.e. we would assume that $\tau_1, \tau_2, \cdots, \tau_n$ and $X_1, X_2, \cdots, X_n$ are already constructed having the desired properties. We now consider the construction of τ_{n+1}.

Let $f_a = \exp(-\lambda\tau_{n+1})I_{[B_n(\tau_{n+1})=a]}$. By assumption $B_n(t)$ is independent of $\mathscr{A}_n$ and, by definition, of τ_{n+1}. $\mathscr{B}(f_a) \subset \mathscr{B}(B_n(t), t \geqq 0)$; therefore $\mathscr{B}(f_a)$ is independent of τ_n. Like Skorohod [**4**, p. 166] let us state a

COROLLARY. *If $B_n(t)$, τ_{n+1} is as above, then for $\lambda>0$ the following hold*:

$$(1) \qquad E(\exp(-\lambda\tau_{n+1}) \mid \mathscr{A}_n) = \frac{\sinh b(2\lambda)^{1/2} - \sinh a(2\lambda)^{1/2}}{\sinh(b-a)(2\lambda)^{1/2}} \quad a.s.;$$

$$(2) \qquad \begin{aligned} E(I_{[B_n(\tau_{n+1})=a]} \mid \mathscr{A}_n) &= b/(b-a) \quad and \\ E(I_{[B_n(\tau_{n+1})=b]} \mid \mathscr{A}_n) &= -a/(b-a); \end{aligned}$$

$$(3) \qquad E(\tau_{n+1} \mid \mathscr{A}_n) = -ab \quad a.s$$

(4) *For each $k>0$, there exists a constant C_k depending on k only such that $E(\tau_{n+1}^k|\mathscr{A}_n) \leqq C_k ab(b-a)^{2k-2}$.*

Here also the σ-field generated by those r.v.'s whose conditional expectation will be constructed is contained in the σ-field $\mathscr{B}(B_n(t), t \geqq 0)$ and hence independent of $\mathscr{A}_n$.

Let $F_n(x, u)$ be the conditional distribution function of $\hat{X}_{n+1}$ given $(X_1, X_2, \cdots, X_n)=x$. If the jumps (or discontinuities) of F are joined by vertical lines and the graph so obtained is reversed then we obtain a function $f_n(x, t)$ on $R^n \times [0, 1]$ to R. For fixed x this $f_n(x, t)$, except for at most countable points t, is defined everywhere in $[0, 1]$. These points are precisely those t's for which $F_n(x, u)=t$. $f_n(x, t)$ is the distribution function mapping. Since $E(\hat{X}_{n+1}|\hat{X}_1, \cdots, \hat{X}_n)=0$, like Skorohod, we get $\int_{\alpha_n(x)}^1 f_n(x, t)\,dt=-\int_0^{\alpha_n(x)} f_n(x, t)\,dt$ where

$$\alpha_n(x) = F_n(x, 0) + \frac{1}{2}\left(\lim_{z\downarrow 0} F_n(x, z) - \lim_{z\uparrow 0} F_n(x, z)\right)$$

and a function $G_n: R^n \times [0, 1]\to[0, 1]$ defined by

$$\int_{\alpha_n(x)}^{y} f_n(x, t)\,dt = -\int_{G_n(x,y)}^{\alpha_n(x)} f_n(x, t)\,dt \quad (\text{if } 1 \geqq y \geqq \alpha_n(x))$$

and

$$\int_{x}^{\alpha_n(x)} f_n(x, t)\,dt = -\int_{\alpha_n(x)}^{G_n(x,y)} f_n(x, t)\,dt \quad (\text{if } 0 \leqq y \leqq \alpha_n(x))$$

and also G_n has the property

$$G_n(x, G_n(x, y)) = x \quad \text{a.s. } (y \in [0, 1]), \qquad G_n(x, \alpha_n(x)) = \alpha_n(x) \quad \text{a.s.}$$

Let Y_{n+1} be the $(n+1)$th uniformly distributed random variable on $(\Omega, \mathscr{B}, P)$. Then we can define two mappings X^a_{n+1} and X^b_{n+1} from $R^n \times [0, 1]$ to R by

$$X^a_{n+1}(x, \omega) = f_n(x, Y_{n+1}(\omega)), \qquad X^b_{n+1}(x, \omega) = f_n(x, G_n(x, Y_{n+1}(\omega))).$$

X^a_{n+1} and X^b_{n+1} are $\mathscr{B}^n \times \mathscr{B}(Y_{n+1})$ measurable random variables where $\mathscr{B}^n$ is the usual Borel field on R^n and $\mathscr{B}(Y_{n+1})$ is the Borel field defined by Y_{n+1}. Like Skorohod [**4**, p. 167] we state the lemma.

LEMMA 3. *Let $B_n(t)$ be independent of $\mathscr{A}_n$ and Y_{n+1}. Let $T(x, \eta)$, $x \in R^n$, $\eta \in [0, 1]$ be the smallest solution of*

$$(B_n(t, \eta) - X^a_{n+1}(x, \eta))(B_n(t, \eta) - X^b_{n+1}(x, \eta)) = 0. \tag{4}$$

Further let $T_{n+1}(\omega)=T(X_1(\omega), \cdots, X_n(\omega), \omega)$ and $X_{n+1}=B_n(T_{n+1})$, then T_{n+1} is $\mathscr{A}_{n+1}$ measurable and $X_1, X_2, \cdots, X_{n+1}$, and $\hat{X}_1, \hat{X}_2, \cdots, \hat{X}_{n+1}$ have the same distribution.

Let F_i be the σ-field generated by $X_1, X_2, \cdots, X_i$ and $F_0=\mathscr{A}_0=\{\phi, \Omega\}$.

PROOF OF THEOREM 1. Now by the Birkhoff ergodic theorem

$$n^{-1}[E(\tau_n \mid \mathscr{A}_{n-1}) + E(\tau_{n-1} \mid \mathscr{A}_{n-2}) + \cdots + E(\tau_2 \mid \mathscr{A}_1) + E(\tau_1 \mid \mathscr{A}_0)]$$

$$\to \frac{1}{n}\sum_{i=1}^{n} E(X_i^2 \mid F_{i-1}) \to EX_1^2 = 1 \quad \text{a.s.};$$

now it is enough to show that

$$\frac{1}{n}\sum_{i=1}^{n} [\tau_i - E(\tau_i \mid \mathscr{A}_{i-1})] \to 0 \quad \text{a.s.}; \tag{5}$$

using Theorem 5 of Chow [1], the result will follow if

$$\sum_{n=1}^{\infty} \frac{1}{n^2} E(\tau_n - E(\tau_n \mid \mathscr{A}_{n-1}))^2 < \infty. \tag{6}$$

Now

$$\sum_{n=1}^{\infty} \frac{1}{n^2} E(\tau_n - E(\tau_n \mid \mathscr{A}_{n-1}))^2 \leqq \sum_{n=1}^{\infty} \frac{1}{n^2} E\tau_n^2 \leqq L_2 \sum_{n=1}^{\infty} \frac{1}{n^2} EX_n^4.$$

Since $(X_i, i \geqq 1)$ is a stationary ergodic sequence with $E(X_i|F_{i-1})=0$ a.s. for all $i \geqq 2$, proceeding as in Skorohod [4] we see that $(\tau_i, i \geqq 1)$ are stationary. Since $(Y_n, n \geqq 1)$ are independent identically distributed uniform random variables on [0, 1] and $(X_n, n \geqq 1)$ are stationary ergodic stochastic sequences, $\{X_n^a\}$ and $\{X_n^b\}$ are also stationary ergodic stochastic sequences.

Now by a truncated argument, e.g., considering $X_n^* = X_n I_{[|X_n| < \varepsilon\sqrt{n}]}$, (6) is true. Therefore $n^{-1}(\tau_1 + \tau_2 + \cdots + \tau_n) \to 1$ a.s. So by Lemma 1,

$$\sup_{\tau \leqq t} \frac{|\tilde{S}_{[\tau]} - B_{(\tau)}|}{(2t \log\log t)^{1/2}} \to 0 \quad \text{a.s. as } t \to \infty.$$

So the theorem follows from the Corollary to Theorem 3 of [6].

Theorem 2. *Strassen's theorem* [6] *holds if $x_1, x_2, \cdots$ are independent r.v.'s with $Ex_i=0$, $Ex_i^2=1$ and*

$$\sum_{k=2}^{\infty} \frac{E|x_k|^{2+\delta}}{k^{1+\delta/2}} < \infty, \qquad 2 \geqq \delta > 0$$

(in particular if $E|x_k|^{2+\delta} \leqq C$ for all k).

Proof. Now

$$E|\tau_{k^{-1}}|^{1+\delta/2} \leqq 2^\delta(E\tau_k^{1+\delta/2} + 1) \leqq 2^\delta(C_\delta E|x_k|^{2+\delta} + 1).$$

So by Theorem 5 of Chow [1]

$$\sum_{k=2}^{\infty} \frac{E|x_k|^{2+\delta}}{k^{1+\delta/2}} < \infty \quad \text{implies} \quad \frac{1}{n}\sum_{1}^{n} (\tau_i - E\tau_i) \to 0 \quad \text{a.s.}$$

Therefore $\{\tau_i\}$ satisfies the conditions of Lemma 1.

Remarks 1. Gaposhkin [2] proved that if $x_1, x_2, \cdots$ are independent r.v.'s with $|x_k| \leqq C$ a.s. and $Ex_k^2=1$, $Ex_k=0$ for all $k=1, 2, \cdots$ then

$$\limsup \sum_{k=1}^{n} \left(1 - \frac{k}{n}\right)^\alpha x_k \left(\frac{2}{2\alpha+1}\, n \log\log n\right)^{-1/2} = 1 \text{ a.s.}, \alpha > 0.$$

We shall extend this result under the conditions of Theorems 1 and 2.

Let $S_0=0$. By Abel's sum

$$\sum_{k=1}^{n}\left(1-\frac{k}{n}\right)^{\alpha}x_k=\sum_{k=0}^{n-1}\left\{\left(1-\frac{k}{n}\right)^{\alpha}-\left(1-\frac{k+1}{n}\right)^{\alpha}\right\}S_k$$
$$\cong\alpha\frac{1}{n}\sum_{k=0}^{n-1}\left(1-\frac{k}{n}\right)^{\alpha-1}S_k.$$

Taking $f(t)=\alpha(1-t)^{\alpha-1}$ on p. 218 of Strassen [6] we get

$$\limsup_{n\to\infty}\sum_{k=1}^{n}\left(1-\frac{k}{n}\right)^{\alpha}x_k(2n\log\log n)^{1/2}$$
$$=\limsup_{n\to\infty}(2n^3\log\log n)^{-1/2}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)S_k=\left(\frac{1}{2\alpha+1}\right)^{1/2}.$$

Similarly Theorem 2 and Theorem 3 of Gaposhkin [2] may be extended.

2. We conjecture that if $x_1, x_2, \cdots$ is a martingale difference sequence with $Ex_1=0$, $Ex_n^2=1$ for all n and $\{x_n^2\}_1^\infty$ is uniformly integrable, then the law of the iterated logarithm holds.

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