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LIE ALGEBRA REPRESENTATIONS OF DIMENSION $p-1$

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ABSTRACT. A semisimple Lie algebra over an algebraically closed field of characteristic $p>2$ admitting a faithful representation of dimension $p-1$ is either a direct sum of classical algebras or the Witt algebra.

I. **Introduction.** In this note we discuss simple Lie algebras of characteristic $p>2$ by use of filtrations. By a filtration of length r of a simple Lie algebra L is meant a sequence of subalgebras

$$L = L_{-1} \supset L_0 \supset \cdots \supset L_r \supset L_{r+1} = (0), \qquad L_r \neq (0),$$

where L_0 is a given proper subalgebra and the remaining algebras are inductively defined by

$$L_{i+1} := \{x \in L_i,\ xL \subset L_i\}.$$

It follows that $L_iL_j \subset L_{i+j}$ for $i+j \geqq -1$.

A filtration is said to be *long* if $r \geqq 2$, *short* if $r=0$, *maximal* if L_0 is a maximal algebra and *nilpotent* if L_0 acts nilpotently on L. Kostrikin calls Lie algebras which possess a long filtration strongly degenerate.

An important result about Lie algebras which possess a long filtration is mentioned in [7] by Kostrikin: There exists an element $c \neq 0$ such that

$$R_c R_{x_1} \cdots R_{x_{p-4}} R_c = 0.$$

Here R_x means the right multiplication: $yR_x := yx$.

Proposition 1 expresses this in terms of the length of possible filtrations.

Using this result and a characterization of Lie algebras which possess no long filtration ([7], [3], [9]), we prove in this note the following

THEOREM. *Let L be a semisimple Lie algebra over an algebraically closed field of characteristic $p>2$ and M a faithful L-module of dimension $p-1$. Then L is either a direct sum of classical algebras or the Witt algebra.*

The representations of the Witt algebra are well known [2]. The possible dimensions of the representation modules are $p-1$ or p^r $(r \geqq 0)$. Semisimple Lie algebras having a representation of lower dimension than $p-1$

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(but >1) are direct sums of simple classical Lie algebras as is shown by J. B. Jacobs [3].

II. **Filtrations.** We use the notation $(xyz)=(xy)z$, $xL^i:=\langle xy_1\cdots y_j, 0\leqq j\leqq i\rangle$, $xL^0:=\langle x\rangle$, where $\langle\ \rangle$ means the linear span. Let L be a simple strongly degenerate Lie algebra. A filtration defined by the proper subalgebra L_0 we denote by $\mathcal{F}(L_0)$ and the length of this filtration by $\delta\mathcal{F}(L_0)$. $\delta(L)$ means the maximum of $\delta\mathcal{F}(L_0)$ for all proper subalgebras. This maximum exists, for L is simple and finite dimensional.

Kostrikin proved the existence of an element $c\neq 0$ for which

$$(*)\qquad \begin{aligned} R_cR_{x_1}\cdots R_{x_{p-4}}R_c &= 0,\\ R_cR_{x_1}\cdots R_{x_{p-3}}R_cR_{y_1}\cdots R_{y_{p-3}}R_c &= 0\end{aligned}$$

hold [6].

PROPOSITION 1. (a) $\delta(L)\geqq p-3$.

(b) *If $\delta(L)=k$, then there exists an element $c\in L$ with $R_cR_{x_1}\cdots R_{x_i}R_c=0$ for all $i\leqq k-1$ if k is even and $i\leqq k-2$ if k is odd.*

PROOF. (a) Using an element c satisfying (*) we define $L_0:=\ker R_c$ This is a proper subalgebra containing (cL^{p-3}); so $\delta\mathcal{F}(L_0)\geqq p-3$.

(b) Let $\mathcal{F}(L_0)$ be a filtration of length k. If $k=2m+1$ then $(cx_0\cdots x_i)\in L_1$ for all $i\leqq k-2$ and any $c\in L_k$. Thus $(cx_0\cdots x_ic)=0$ and $R_cR_{x_1}\cdots R_{x_i}R_c=0$. If $k=2m$, $c\in L_k$, then

$$((cx_1\cdots x_i)(cy_1\cdots y_j))\in L_{k-i+k-j}=(0)\quad \text{if } i+j<k.$$

Therefore it holds that

$$0=((cx_1\cdots x_m)(cx_1\cdots x_m))=(-1)^m(cx_1^2\cdots x_m^2c).$$

By linearization it follows that $(cy_1\cdots y_{2m}c)=0$ for all $y_i\in L$.

PROPOSITION 2. *There exists a nilpotent filtration of length $\geqq p-3$.*

PROOF. Using an element c satisfying (*) we define

$$L_0:=(cL^{p-3})\subset \ker R_c.$$

L_0 is a proper subalgebra, and $\delta\mathcal{F}(L_0)\geqq p-3$. We prove that L_0 acts nilpotently on L:

$$\begin{aligned}&R(cx_1^{p-3})\cdots R(cx_n^{p-3})\\ &\quad=\sum\alpha(i_1,\cdots,i_n)R(x_1)^{i_1}R(c)R(x_1)^{p-3-i_1}R(x_2)^{i_2}R(c)\cdots R(c)R(x_n)^{p-3-i_n}.\end{aligned}$$

If $p-3-i_j+i_{j+1}\leqq p-3$ and $p-3-i_{j+1}+i_{j+2}\leqq p-3$ for some j then this summand equals 0 by (*). Thus assume $i_{j+2}>i_j$ for all $j=1,\cdots,n-2$. It follows that $i_{2r+1} \geqq r$ which is impossible if $n > 2(p-2)$. By linearization the result follows.

The natural number $\delta(L)$ is closely connected with the concept of height of a Lie algebra mentioned by Kostrikin [6]. The height $\Delta(L, x)$ of L with respect to x is defined to be the minimal index n for which $(xL^n)=L$. Then $\Delta(L)$, the maximum of $\Delta(x, L)$ over all x in L, he called the height of the simple Lie algebra L. We get a filtration of length $\Delta(L, x)-2$ by the following construction.

Define $L_0 := \{z \in L \mid (zL) \subset xL^{\Delta(L,x)-1}\}$. If $z_1, z_2 \in L_0$ then

$$((z_1 z_2)u) = ((z_1 u)z_2) + (z_1(z_2 u)) \subset xL^{\Delta(L,x)-1} \neq L.$$

Thus L_0 is a proper subalgebra defining a filtration of length $\Delta(L, x)-2$. If, on the other hand, $\delta\mathscr{F}(L_0)=k$ and $c \in L_k$, then $(cL^k) \subset L_0$ and $\Delta(L, c) \geqq k+1$ holds. This proves

PROPOSITION 3. $\Delta(L) \geqq \delta(L)+1 \geqq \Delta(L)-1$.

$\Delta(L)=\delta(L)+1$ (equal to $p-1$) holds for the Witt algebra, $\Delta(L)=\delta(L)+2$ (equal to $2p-2$) for the Block algebra $L(G_0, 0, f)$ of dimension p^2-1. Kostrikin stated [6] that there exists a fixed ρ such that $\Delta(L) \leqq \rho$ for all classical simple Lie algebras. The main result [7] of Kostrikin and the extensions by J. B. Jacobs [3] and the author [9] can be combined to read:

$\delta(L)=1$ if and only if L is classical, provided the ground field is algebraically closed (thus $\Delta(L) \leqq 3$ in this case).

It is an open question whether there exist Lie algebras for which $\delta(L)=0$ holds. We need a characterization of these algebras obtained in [9].

PROPOSITION 4. *Let L be a simple Lie algebra of characteristic $p>3$. Then $\delta(L)=0$ if and only if $R_x^{p-1} \neq 0$ for all $x \in L$.*

III. **Proof of the theorem.** The proof is done by several lemmas. If $p=3$ an elementary computation shows that L is the three-dimensional simple Lie algebra. So we assume $p>3$. An argument of J. B. Jacobs [3] based upon a result of Block on semisimple Lie algebras [1] ensures

$$\bigoplus \operatorname{der} S_i \supset L \supset \bigoplus \operatorname{ad} S_i$$

where S_i are simple Lie algebras and der S_i denotes the derivation algebra of S_i. Let $S_i := S$. We claim S possesses a faithful irreducible representation module N of dimension $\leqq p-1$. If not, there exists $m \in M$ such that $mS=(0)$. M/Km is a faithful module of lower dimension. By induction it follows that S is nilpotent, a contradiction. By T we denote the representation $S \to \operatorname{End}(N)$.

LEMMA 1. *For y in L, $T(y)$ is nilpotent if and only if R_y is nilpotent. If either (and then both) is nilpotent, then $R_y^p = T(y)^{p-1} = 0$.*

PROOF. $T(y)$ nilpotent implies $T(y)^{p-1}=0$. It follows that $uT((zy^p))=uT(z)T(y)^p-uT(y)^pT(z)=0$ for all $z\in L$, $u\in N$, whence $R_y^p=0$.

Let $R(y)$ be nilpotent, that is $(Ly^{p^r})=0$. Then $T^{p^r}(y)=\lambda\,\mathrm{Id}_N$ because N is irreducible. $T(y)$ has only one eigenvalue μ and $0=\mathrm{tr}(T(y))=\mu\dim N$. Thus $\lambda=\mu^{p^r}=0$, $T(y)$ is nilpotent. $T^{p-1}(y)=0$ holds for $\dim N\leqq p-1$.

(1) Now we assume $\delta(S)=0$.

LEMMA 2. (a) *Let $T(y)$ be nilpotent. Then it follows that $NT^k((ay^{p-1}))\subset NT^k(y)$ for all $a\in S$, $k=1,\cdots,p-1$.*

(b) *If $T^{p-2}(y)=0$ then $y=0$.*

PROOF. $T^k(ay^{p-1})=\sum\alpha((i_1,\cdots,i_{k+1}))T(y)^{i_1}T(a)T(y)^{i_2}\cdots T(a)T(y)^{i_{k+1}}$. If $i_j\geqq p-1$ for some i this term vanishes. So we assume $i_j\leqq p-2$ for all j. We have $k(p-1)=i_{k+1}+\sum_{j=1}^k i_j\leqq(p-2)k+i_{k+1}$, that is $i_{k+1}\geqq k$. If $T^{p-2}(y)=0$ we can assume $i_j\leqq p-3$. It follows that $(k=(p-1)/2)$

$$\left(\frac{p-1}{2}\right)(p-1)=i_{(p+1)/2}+\sum_{j=1}^{(p-1)/2}i_j\leqq i_{(p+1)/2}+(p-3)\left(\frac{p-1}{2}\right),$$

that is, $i_{(p+1)/2}\geqq p-1$, and $T^{(p-1)/2}((ay^{p-1}))=0$ for all $a\in S$. Now $R_y^{p-1}\neq 0$ if $y\neq 0$ (for $\delta(L)=0$). There exists $a\in S$ such that $z:=ay^{p-1}\neq 0$. We have

$$uT((bz^{p-1}))=\sum_{i=0}^{p-1}uT^i(z)T(b)T^{p-1-i}(z)=0\quad\text{for all } u\in N,\ b\in S.$$

From this it follows that $(bz^{p-1})=0$ for all $b\in S$, so $z=0$. This contradiction proves the lemma.

Now let H be a Cartan subalgebra of S and $S=\sum S_\alpha$, $N=\sum N_\beta$ decompositions with respect to R_H and $T(H)$.

If $\alpha\neq 0$ is any root and $x\in S_\alpha$, by Lemma 2, $NT^{p-2}(x)\neq 0$. There exists $u\in N$ such that $uT^{p-2}(x)\neq 0$. Then the $N_{\beta+i\alpha}$ $(0\leqq i\leqq p-2)$ are $(p-1)$ different weight spaces. It follows that $\dim N_{\beta+i\alpha}=1$. Let $h\in H$. Then

$$0=\mathrm{tr}(T(h))=\sum_{i=0}^{p-2}(\beta+i\alpha)(h)=(p-1)\beta(h)+\alpha(h),$$

whence $\alpha=\beta$.

We proved $N_\alpha T(x)\neq 0$ for all $x\in S_\alpha$ and roots $\alpha\neq 0$. But then $\dim S_\alpha=1$ is true because all weight spaces of N are one dimensional. If $\dim H\neq 1$ there exists $h\in H$ such that $\alpha(h)=0$. It then holds that $T(h)=0$, a contradiction. Therefore $\dim S=p$ holds and S is the Witt algebra, $\delta(S)\neq 0$.

(2) S is classical if $\delta(S)=1$ and $p>5$. If $p=5$ a detailed computation shows that S has nondegenerate trace $\mathrm{tr}(T(x)T(y))$, and thus is classical. From $\dim N<p$ and **[8]** follows that S has nondegenerate trace form in each case. This implies $\mathrm{der}\,S=\mathrm{ad}\,S$.

(3) We assume $\delta(S)=t>1$. By Proposition 2 there exists a nilpotent filtration $\mathscr{F}(S_0)$ of length $p-3$ if $t=p-3$ and of length $t-1$ if $t>p-3$. From Lemma 1 it follows that $T(S_0)$ is a Lie algebra of nilpotent transformations on N. There exists $m \in N$, $m\neq 0$, for which $mT(S_0)=0$. We define an ascending chain of vector spaces by

$$N_0:=Km, \qquad N_i:=\langle mT(x_1)\cdots T(x_j), 0\leqq j\leqq i, x_j\in S\rangle.$$

N is irreducible, so $N_k=N_{k+1}=N$ for some k.

LEMMA 3. (a) $N_jT(S_{p-3-i})=(0)$ *if* $i+j\leqq p-3$.
(b) $N_k\neq N$ *if* $k<p-2$.
(c) $\dim N_i/N_{i+1}=1$, $0\leqq i\leqq p-2$.
(d) *There exists* $x\in S$ *such that* $N_i=\sum_{j=0}^{i} KmT^j(x)$.
(e) $\delta\mathscr{F}(S_0)\leqq p-3$, $\delta(S)\leqq p-2$.

PROOF. (a) follows easily by induction on j.
(b) $N_kT(S_{p-3})=(0)$ if $k<p-2\Rightarrow N_k\neq N$.
(c) is a direct consequence of (b) because $\dim N\leqq p-1$.
(d) If $mT^{p-2}(y)\in N_{p-3}$ for all $y\in S$, then by linearization $N_{p-2}\subset N_{p-3}$ holds, a contradiction. So there exists $x\in S$ such that $N_{i-1}\not\supset\langle mT^i(x)\rangle$ $(0\leqq i\leqq p-2)$.
(e) If $c\in S_{p-2}$, then $mT((cx^i))=0$ holds for all $i\leqq p-2$. It follows that $NT(c)=(0)$ and $c=0$. The construction of $\mathscr{F}(S_0)$ implies $\delta(S)\leqq(p-3)+1$.
Now we can prove

PROPOSITION 5. *S is the Witt algebra.*

PROOF. Define $S_0':=\{z\in S \mid mT(z)\in Km\}$. S_0' is a proper subalgebra of S containing S_0. If $y\in S\backslash S_0'$, then $mT(y)\in N_1=Km+KmT(x)$. It follows that $mT(y)=\alpha(y)m+\beta(y)mT(x)$ and $y-\beta(y)x\in S_0'$. So S_0' defines a filtration $\mathscr{F}(S_0')$ of length $\leqq p-2$ for which $\dim S/S_0'=1$. If $e_1,\cdots,e_r\in S_i'$ are linearly independent modulo S_{i+1}' then $(e_1x),\cdots,(e_rx)$ are linearly independent modulo S_i' by definition of filtrations. This means

$$\dim S_i'/S_{i+1}'\leqq\dim S_{i-1}'/S_i'\leqq\cdots\leqq\dim S/S_0'=1.$$

It follows that $\dim S=\delta\mathscr{F}(S_0')+2\leqq p$. One easily proves from this that S is the Witt algebra.

The equality $\operatorname{der} S=\operatorname{ad} S$ holds for the Witt algebra. As the conclusion of all this we get $L=\bigoplus S_i$, where S_i is either classical or the Witt algebra. Now assume S_1 to be the Witt algebra. We show $L':=\bigoplus_{i>1}S_i=(0)$. There is only one representation of S_1 of dimension $p-1$, but none of lower dimension. If we take a canonical basis (e_i), $-1\leqq i\leqq p-2$, satisfying $e_ie_j=(j-i)e_{i+j}$ (or $=0$ if $i+j>p-2$), then there exists $u\in M$ such that $M=\sum_{i=0}^{p-2}KuT^i(e_{-1})$, $uT^{p-1}(e_{-1})=0$ and $vT(e_1)=0$ if and only if $v\in Ku$.

This follows from [2]. From $uT(L')T(e_1)=uT(e_1)T(L')=0$ we have that $uT(L')\subset Ku$ and $uT(L')=(0)$ (for $L'=L'L'$). This leads to $uT^i(e_{-1})T(L')=uT(L')T^i(e_{-1})=0$ for all i and $L'=(0)$. Q.E.D.

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