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AN INVERSE FUNCTION THEOREM FOR FREE GROUPS

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ABSTRACT. Let F_n be a free group of rank n with free basis $x_1, \cdots, x_n$. Let $\{y_1, \cdots, y_k\}$ be a set of $k \leqq n$ elements of F_n, where each y_i is represented by a word $Y_i(x_1, \cdots, x_n)$ in the generators x_j. Let $\partial y_i/\partial x_j$ denote the free derivative of y_i with respect to x_j, and let $J_{kn} = \|\partial y_i/\partial x_j\|$ denote the $k \times n$ Jacobian matrix. THEOREM. *If $k=n$, the set $\{y_1, \cdots, y_n\}$ generates F_n if and only if J_{nn} has a right inverse. If $k<n$, the set $\{y_1, \cdots, y_k\}$ may be extended to a set of elements which generate F_n only if J_{kn} has a right inverse.* Several applications are given.

Let Z denote the ring of rational integers, and let ZF_n denote the integral group ring of a free group F_n which has the free basis $x_1, \cdots, x_n$. Let $\partial/\partial x_j : ZF_n \to ZF_n$ denote the jth free partial derivative, in the sense of R. H. Fox [3]. The mapping $\partial/\partial x_j$ is defined as follows: If $w \in F_n$ is represented by the word $x_{\mu_1}^{\varepsilon_1} \cdots x_{\mu_r}^{\varepsilon_r}$, where $\varepsilon_i = \pm 1$ and $\mu_i = 1, \cdots, n$, then $\partial w/\partial x_j \in ZF_n$ is defined by:

$$\frac{\partial w}{\partial x_j} = \sum_{k=1}^{n} \delta_{j,\mu_k} \varepsilon_k x_{\mu_1}^{\varepsilon_1} \cdots x_{\mu_{k-1}}^{\varepsilon_{k-1}} x_{\mu_k}^{(\varepsilon_k - 1)/2} \tag{1}$$

where δ means the Kronecker symbol. More generally, if $u = \sum_{i=1}^{t} c_i w_i$, $c_i \in Z$, $w_i \in F_n$, we define

$$\frac{\partial u}{\partial x_j} = \sum_{i=1}^{t} c_i \frac{\partial w_i}{\partial x_j}. \tag{2}$$

It is easy to show that this definition is independent of the choice representatives of the elements $w_i \in F_n$.

There are known analogues between the "free" calculus of polynomials in the noncommuting indeterminates $x_1, \cdots, x_n$ and the "ordinary" calculus of polynomials in commuting indeterminates, such as the existence of Taylor series [3]. However, it is worth noting that if α is the

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abelianizing homomorphism acting on F_n, and if α_* is the induced homomorphism from $ZF_n \to ZF_n^{\alpha}$, then the image under α_* of the free partial derivative of u, as defined by (2), is not the ordinary partial derivative of $\alpha_*(u)$. Thus the free calculus and the ordinary calculus appear to be distinct theories. The purpose of this note is to point out a new analogue between the free calculus and the ordinary calculus, an "inverse function theorem" for free groups.

INVERSE FUNCTION THEOREM. *Let $\{y_1, \cdots, y_k\}$ be a set of $k \leq n$ elements of F_n. Let J_{kn} denote the $k \times n$ "Jacobian" matrix $\|\partial y_i/\partial x_j\|$.*

(i) *If $k=n$, a necessary and sufficient condition for $\{y_1, \cdots, y_n\}$ to be a generating set for F_n is that J_{nn} have a right inverse.*

(ii) *If $k<n$, a necessary condition for $\{y_1, \cdots, y_k\}$ to extend to a generating set $\{y_1, \cdots, y_n\}$ is that J_{kn} have a right inverse.*

PROOF. We first establish the sufficiency of the condition (i).[2] Suppose that $B = \|\beta_{ij}\|$ is a right inverse of J_{nn}. By a theorem of M. S. Montgomery [6], the matrix B is also a left inverse of J_{nn}. Hence

$$\sum_{s=1}^{n} \beta_{is}\left(\frac{\partial y_s}{\partial x_j}\right) = \delta_{ij} \qquad (i, j = 1, \cdots, n). \tag{3}$$

Multiplying both sides of (3) by $x_j - 1$, and summing over j, we obtain

$$\sum_{s=1}^{n} \beta_{is} \sum_{j=1}^{n} \frac{\partial y_s}{\partial x_j}(x_j - 1) = x_i - 1 \qquad (i = 1, \cdots, n). \tag{4}$$

By the "fundamental theorem" of free calculus [3]:

$$\sum_{j=1}^{n} \frac{\partial y_s}{\partial x_j}(x_j - 1) = y_s - 1 \qquad (s = 1, \cdots, n). \tag{5}$$

Hence

$$\sum_{s=1}^{n} \beta_{is}(y_s - 1) = x_i - 1 \qquad (i = 1, \cdots, n). \tag{6}$$

Now let H be the subgroup of F_n generated by $y_1, \cdots, y_n$ and let I_H be the ideal of ZF_n generated by $y_1 - 1, \cdots, y_n - 1$. According to equation (6), the ring elements $x_i - 1$ belong to I_H for each $i = 1, \cdots, n$. But then, by Lemma 4.1 of [2], it follows that $x_i \in H$ for each $i = 1, \cdots, n$. Hence H coincides with F_n, and our result is established.

Necessity may be established by noting that if $\{y_1, \cdots, y_k\}$ extends to a basis $\{y_1, \cdots, y_n\}$, then we may write each x_i as a word $X_i(y_1, \cdots, y_n)$,

[2] The very brief proof of sufficiency given here was suggested by the referee. It replaces a longer and more computational proof in an earlier version of this paper. The author wishes to thank the referee for his constructive suggestions.

in the generators y_j. Moreover, if y_i is represented by the word $Y_i(x_1, \cdots, x_n)$, then we will have

$$(7)\quad Y_i(X_1(y_1, \cdots, y_n), \cdots, X_n(y_1, \cdots, y_n)) = y_i \qquad (i = 1, \cdots, n).$$

The chain rule (see [3]) applied to (7) gives

$$(8)\quad \sum_{s=1}^{n} \left(\frac{\partial Y_i(x_1, \cdots, x_n)}{\partial x_s}\right)\left(\frac{\partial X_s(y_1, \cdots, y_n)}{\partial y_j}\right) = \delta_{ij} \qquad (i, j = 1, \cdots, n).$$

Hence the $n \times n$ matrix J_{nn} has a right inverse, $J_{nn}^* = \| \partial x_i / \partial y_j \|$. Thus condition (i) is seen to be sufficient. Also the $k \times n$ submatrix J_{kn} formed by the first k rows of J_{nn} has a right inverse, namely the submatrix J_{nk}^* of J_{nn}^* formed by the first k columns of J_{nn}^*. This proves (ii).

As an application, we will use the inverse function theorem to give a new proof of a classical theorem of J. Nielsen. This application was suggested by the referee. The proof below is a modification of his proof.

COROLLARY 1 (J. Nielsen, see [5]). *Any set of n elements which generate a free group of rank n are a set of free generators.*

PROOF. Suppose that $\{y_1, \cdots, y_n\}$ generate F_n, and suppose also that $\{y_1, \cdots, y_n\}$ satisfy the relation

$$(9)\qquad r(y_1, \cdots, y_n) = 1,$$

where we assume that $r(y_1, \cdots, y_n)$ is freely reduced as a word in the y_j's. By the chain rule it follows that

$$(10)\qquad \sum_{\partial=1}^{n} \frac{\partial r}{\partial y_j} \frac{\partial y_j}{\partial x_i} = 0, \qquad i = 1, \cdots, n.$$

Setting $\mathbf{r} = (\partial r / \partial y_1, \cdots, \partial r / \partial y_n)$, we may rewrite (10) in the form

$$(11)\qquad \mathbf{r} J_{nn} = \mathbf{o}.$$

By the inverse function theorem, the matrix J_{nn} has a right inverse, say B. Multiplying both sides of (11) by B we obtain

$$(12)\qquad \mathbf{r} J_{nn} B = \mathbf{r} I = \mathbf{r} = \mathbf{o},$$

hence $\partial r / \partial y_j = 0$ for each $j = 1, \cdots, n$. But then it follows that r does not involve the letter y_j, since no cancellations are possible in (1) if a word is freely reduced. It then follows that r must be the trivial relator. This completes the proof of Corollary 1.

We observe that our theorem may be applied both ways. Consider first the case $k = n$. An algorithm for deciding whether a set of n elements in a free group F_n are a basis was discovered by J. Nielsen (see Chapter 2 of **[5]**), and Nielsen's algorithm translates into a straightforward algorithm for expressing a Jacobian matrix as a product of elementary invertible matrices over ZF_n.[3] On the other hand, it is known [7] that the ring ZF_n can be embedded in a skew field K. Since a procedure exists for finding inverses of invertible matrices over skew fields (see Chapter IV of **[1]**), and since such inverses are unique, one may decide whether a Jacobian matrix is invertible over ZF_n by computing its inverse over K, and seeing whether in fact the entries are in ZF_n. This yields a new test to decide if $y_1, \cdots, y_n$ are a basis. This procedure is not however, a practical alternative to Nielsen's relatively simple algorithm; the elementary invertible matrices which are obtained by the method in **[1]** are almost certainly not in ZF_n, even when their product is in ZF_n, so that this procedure is unnecessarily complex. Our theorem does, however, yield a very simple necessary condition which a set $\{y_1, \cdots, y_n\}$ must survive if it is a basis:

COROLLARY 2. *Let J^α_{nn} denote the image of J_{nn} under the abelianizing homomorphism α_* acting on ZF_n. Then $\{y_1, \cdots, y_n\}$ is a basis for F_n only if* $\det J^\alpha_{nn}$ *is a unit in ZF^α_n.*

PROOF. By Theorem 1, p. 59, of **[4]**, a square matrix over a commutative ring with 1 is invertible if and only if its determinant is a unit. □

To see that the condition in Corollary 2 is not sufficient, let $n=2$ and consider the elements

$$y_1 = x_1, \tag{13}$$

$$y_2 = x_2x_1x_2x_1^{-1}x_2^{-1}x_1^2x_2^2x_1^{-2}x_2^{-1}x_1x_2^{-1}x_1^{-1}x_2^2x_1^2x_2^{-2}x_1^{-2}. \tag{14}$$

A simple calculation shows that $\det\|\partial y_i/\partial x_j\|^\alpha=1$, yet y_1 and y_2 are not primitive, by the test given in Corollary N4, p. 169, of **[5]**.

The more difficult question of deciding whether a set of $k<n$ elements in a free group are primitive was solved by J. H. C. Whitehead **[9]**, **[10]** and by E. Rapaport **[8]**, and the inverse function theorem may be applied to yield an analogous algorithm for deciding when a $k\times n$ Jacobian matrix over ZF_n has a right inverse. Once again, the Jacobian matrices corresponding to Whitehead transformations are a very pleasant set of elementary invertible matrices.

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[3] Note that the mapping which we have defined from Aut F_n to the ring of invertible matrices over ZF_n is a crossed homomorphism. That is, if α and β are automorphisms of F_n which have Jacobian matrices $\|a_{ij}\|$ and $\|b_{ij}\|$ respectively, then the Jacobian matrix corresponding to $\alpha\beta$ is the product $\|a_{ij}\|\ \|b_{ij}\|_\alpha$, where $\|b_{ij}\|_\alpha$ denotes the Jacobian matrix of β with respect to the transformed basis $\alpha(x_1), \cdots, \alpha(x_n)$. Thus, in order to apply Nielsen's algorithm, one must repeatedly change basis.

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