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REAL HOMOGENEOUS ALGEBRAS

D. Ž. DJOKOVIĆ[1]

ABSTRACT. Let (A, μ) be a finite dimensional real algebra (not necessarily associative) with multiplication $\mu \neq 0$. Assuming that $\mathrm{Aut}(A)$ is transitive on one-dimensional subspaces we determine all such algebras. There are up to isomorphism only four such algebras, one in each of the dimensions 1, 3, 6, 7.

Introduction. For the terminology we refer to Bourbaki [3]. All the algebras considered in this paper are assumed to be *finite dimensional.* Let A be an algebra over a field F, $\mu: A \otimes A \to A$ its multiplication and $\mathrm{Aut}(A)$ the group of algebra automorphisms of A. We shall say that A is *homogeneous* if $\mathrm{Aut}(A)$ is transitive on one-dimensional subspaces of A, we shall say that A *is extremely homogeneous* if $\mathrm{Aut}(A)$ is transitive on $A \backslash \{0\}$. Kostrikin [7] has shown that if char $F \neq 2$, A extremely homogeneous and $\mu \neq 0$, then F must be a finite field. On the other hand Shult [11] has shown that if A is homogeneous, $F = \mathrm{GF}(q)$, $q > 2$ and $\mu \neq 0$ then $A \cong F$. The case $F = \mathrm{GF}(2)$ has been considered by Gross [4]. Świerczkowski has shown [13] that when $F = \boldsymbol{R}$ (the real field) and A is a homogeneous Lie algebra with $\mu \neq 0$ then A is isomorphic to the Lie algebra of skew-symmetric 3×3 real matrices. Many of these results have been improved by Mr. L. Sweet [12]. In particular, he has determined all two-dimensional homogeneous algebras and has shown that there are no nontrivial homogeneous algebras over an algebraically closed field.

In this paper we shall determine all real homogeneous algebras. If A is an F-algebra and $B \subset A$ a subspace we define a multiplication in B by choosing a vector space complement C for B in A and putting

$$\mu_B(b_1 \otimes b_2) = \pi \mu_A(b_1 \otimes b_2)$$

where $\pi: A \to B$ is the projection with kernel C. We say then that (B, μ_B) is obtained from (A, μ_A) by truncation. Note that the definition of μ_B depends on the choice of C.

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We can apply this to quaternions $\boldsymbol{H}$ by choosing B to be the subspace of pure quaternions and $C=\boldsymbol{R}\cdot 1$. The corresponding truncated algebra $\bar{\boldsymbol{H}}$ will be called *the algebra of pure quaternions*; it is isomorphic to the three-dimensional Lie algebra mentioned above. Similarly, if we take $A=\boldsymbol{O}$ the algebra of octonions, choose B to be the subspace of pure octonions and $C=\boldsymbol{R}\cdot 1$ then the corresponding truncated algebra $\bar{\boldsymbol{O}}$ will be called *the algebra of pure octonions*. It is well known that these two algebras are homogeneous.

Let $\boldsymbol{T}=\boldsymbol{C}^3$ considered as a real vector space and define the multiplication in $\boldsymbol{T}$ as follows: If $x, y\in \boldsymbol{T}$ and

$$x=\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix},\qquad y=\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix},$$

then

$$x\cdot y=\begin{pmatrix}\bar{\xi}_2\bar{\eta}_3-\bar{\xi}_3\bar{\eta}_2\\ \bar{\xi}_3\bar{\eta}_1-\bar{\xi}_1\bar{\eta}_3\\ \bar{\xi}_1\bar{\eta}_2-\bar{\xi}_2\bar{\eta}_1\end{pmatrix}.$$

Then $\boldsymbol{T}$ is a homogeneous algebra over reals of dimension 6. The group SU(3) acts on $\boldsymbol{T}$ by matrix multiplication and these multiplications are automorphisms of the algebra $\boldsymbol{T}$.

Result and proof. Let (A, μ) be a homogeneous F-algebra, $\mu\neq 0$ and $G=\mathrm{Aut}(A)$. Then A is irreducible as a G-module. Let Γ be any subgroup of G such that A is an irreducible Γ-module. The multiplication $\mu: A\otimes A\to A$ is a homomorphism of Γ-modules. If char $F=0$ then $A\otimes A$ is a semisimple Γ-module (see [**5**, p. 85]). Thus A has to be isomorphic as a Γ-module to a direct summand of $A\otimes A$.

THEOREM. *If (A, μ) is a real homogeneous algebra and $\mu\neq 0$ then it is isomorphic to $\boldsymbol{R}$, $\bar{\boldsymbol{H}}$, $\boldsymbol{T}$ or $\bar{\boldsymbol{O}}$.*

PROOF. If dim $A=n=1$ this is clear. Let $n\geqq 2$ and identify the sphere S^{n-1} with the manifold of oriented one-dimensional subspaces of A. We have the canonical maps $A\backslash\{0\}\to S^{n-1}\to \boldsymbol{P}(A)$ where $\boldsymbol{P}(A)$ is the associated projective space. The action of G on A induces an action on S^{n-1} and $\boldsymbol{P}(A)$. The action of G on $\boldsymbol{P}(A)$ is transitive since A is a homogeneous algebra. Since G is a real Lie group its identity component G_0 is also transitive on $\boldsymbol{P}(A)$ and S^{n-1}.

If $n\geqq 3$ then S^{n-1} is simply connected and, by a result of Montgomery [**9**], G_0 has a compact subgroup Γ which is also transitive on S^{n-1}. By taking a subgroup of Γ (if necessary) we can assume in addition that it is a simple Lie group (see [**8**]). The possible groups Γ which satisfy all these conditions have been determined by Montgomery and Samelson [**8**] and Borel [**1**] and [**2**]. They are as follows:

SO(n), SU(m) for $n=2m$,

SP(m) for $n=4m$, Spin(9) for $n=16$,

Spin(7) for $n=8$, G_2 for $n=7$.

Also, the actions of these groups on S^{n-1} are equivalent to the orthogonal actions obtained by embedding SO(n), SU(m), Sp(m) in SO(n) in the usual way; G_2 is embedded in SO(7) as the automorphism group of $\boldsymbol{O}$ restricted to $\bar{\boldsymbol{O}}$. The embeddings of Spin(9) and Spin(7) are given by the real spin representations Δ_9 and Δ_7. These results have been proved by Poncet [**10**].

Since SU$(m)\subset$SO(n) $(n=2m)$ and Sp$(m)\subset$SU$(2m)$ $(n=4m)$ we can reduce the above list to the following

(i) n odd, $\Gamma=$SO(n) or G_2 (if $n=7$);

(ii) $n=2k$, k odd, $\Gamma=$SU(k);

(iii) $n=4k$, $\Gamma=$Sp(k) or Spin(9) (if $n=16$) or Spin(7) (if $n=8$).

We shall now consider each of these possibilities.

(i) If $\Gamma=$SO(n) then $A\otimes A$ decomposes into direct sum of the symmetric and the skew-symmetric parts. The skew-symmetric part is an irreducible Γ-module. The symmetric part decomposes into two irreducible summands one of them being the trivial module. By comparison of the dimensions we see that A is not isomorphic as a Γ-module to any of the summands in $A\otimes A$ if $n\geqq 5$. In the case $n=3$, A is isomorphic to the skew-symmetric part of $A\otimes A$. Thus there exists a nonzero Γ-homomorphism $\mu: A\otimes A\to A$.

Since A is absolutely irreducible as a Γ-module it follows that μ is unique up to a scalar factor. This means that the algebra (A, μ) is unique up to isomorphism. It is clear that this is the algebra of pure quaternions $\bar{\boldsymbol{H}}$.

Now, assume that Γ is the simple real Lie group G_2 and $n=7$. Consider the corresponding representation of the real Lie algebra G_2 in A. We shall use the notation of Jacobson [**6**, Chapter VII, Theorem 9] to denote its highest weight by λ_2. The weights of this representation are all simple and they are

$$\lambda_2,\quad \lambda_1-\lambda_2,\quad -\lambda_1+2\lambda_2,\quad \lambda_1-2\lambda_2,\quad -\lambda_1+\lambda_2,\quad -\lambda_1,\quad 0.$$

The representation $(\lambda_2)\otimes(\lambda_2)$ decomposes into symmetric and skew-symmetric parts. An irreducible representation of highest weight $2\lambda_2$ is contained in the symmetric part of $(\lambda_2)\otimes(\lambda_2)$. The dimension of $(2\lambda_2)$ is 27 and the dimension of the symmetric part of $(\lambda_2)\otimes(\lambda_2)$ is 28. Hence, the symmetric part of $(\lambda_2)\otimes(\lambda_2)$ decomposes as $(2\lambda_2)+(0)$ where (0) stands for the trivial irreducible representation (of dimension 1).

Similarly, by analyzing the weights of the skew-symmetric part of $(\lambda_2)\otimes(\lambda_2)$ we find that it decomposes as $(\lambda_1)\oplus(\lambda_2)$.

Thus we have $(\lambda_2)\otimes(\lambda_2)=(2\lambda_2)\oplus(0)\oplus(\lambda_1)\oplus(\lambda_2)$.

We have an analogous decomposition of $A\otimes A$ as a Γ-module. Hence there exists a nonzero Γ-homomorphism $\mu: A\otimes A\to A$. Again A is absolutely irreducible as a Γ-module and consequently μ is unique up to a scalar factor. This means that the algebra (A, μ) is unique up to isomorphism. Of course, this algebra is isomorphic to $\bar{O}$ the algebra of pure octonions.

(ii) Now let $\Gamma=\mathrm{SU}(k)$, $n=2k$. Then A can be equipped with a complex structure so that it becomes complex Γ-module of complex dimension k. We shall denote this complex Γ-module by A_C. Moreover, there is a positive definite hermitian form $\langle x, y\rangle$ on A_C which is preserved by Γ. We shall also consider A_C as a complex module over the Lie algebra $L=\mathrm{su}(k)$ of $\mathrm{SU}(k)$. Then of course, A_C can be considered also as a module over the complexified Lie algebra $L_C=\mathrm{sl}(k, C)$. The real L-module A is obtained from A_C by restriction of scalars.

We are interested in analyzing the L-module $A\otimes_R A$. By complexification we get the L_C-module $(C\otimes_R A)\otimes_C (C\otimes_R A)$. We have $C\otimes_R A= M\oplus N$, where

$$M=\{1\otimes x - i\otimes ix \mid x\in A\}, \qquad N=\{1\otimes x + i\otimes ix \mid x\in A\},$$

are L_C-submodules. The map

$$A_C\to M, \qquad x\mapsto 1\otimes x - i\otimes ix,$$

is an isomorphism of L_C-modules. The bilinear form ϕ on $C\otimes_R A$ which sends $(\alpha\otimes x, \beta\otimes y)\mapsto \alpha\beta\langle x, y\rangle$ is L_C-invariant and nondegenerate. If we agree that $\langle x, y\rangle$ is linear in the first variable then the restriction of ϕ to $M\times N\to C$ is also nondegenerate since

$$\phi(1\otimes x - i\otimes ix, 1\otimes y + i\otimes iy) = 4\langle x, y\rangle.$$

Hence the L_C-module N is isomorphic to the contragredient of M, i.e., $C\otimes_R A\cong A_C\oplus A_C^*$. It follows that the complexification of $A\otimes_R A$ is isomorphic to

$$(A_C\otimes_C A_C)\oplus(A_C^*\otimes_C A_C^*)\oplus 2(A_C\otimes_C A_C^*)$$

as an L_C-module.

The highest weight of A_C is λ_1 and that of A_C^* is λ_{k-1}. Here we use again the notation of Jacobson [**6**, Chapter VII, Theorem 6]. By an easy computation we find that

$$(\lambda_1)\otimes(\lambda_1) = (2\lambda_1)\oplus(\lambda_2),$$
$$(\lambda_{k-1})\otimes(\lambda_{k-1}) = (2\lambda_{k-1})\oplus(\lambda_{k-2}),$$
$$(\lambda_1)\otimes(\lambda_{k-1}) = (\lambda_1+\lambda_{k-1})\oplus(0),$$

where (λ) denotes the isomorphism class of an irreducible L_C-module of highest weight λ.

If $k>3$ then we see from these decompositions that A cannot be a direct summand of $A\otimes A$.

When $k=3$ we see that $A\otimes A$ contains A as a direct summand with multiplicity one. Hence, there exists a nonzero Γ-homomorphism $\mu:A\otimes A\to A$ and the algebra (A, μ) is homogeneous and has dimension 6. The endomorphism ring of the Γ-module A is the complex field $\boldsymbol{C}$ and it follows easily that (A, μ) must be isomorphic to the algebra $\boldsymbol{T}$ described in the introduction.

(iii) In all these cases $-I\in\Gamma$ and $(-I)\otimes(-I)$ is the identity map on $A\otimes A$. Hence $A\otimes A$ is not faithful as a Γ-module and A cannot be isomorphic to a direct summand of $A\otimes A$.

It remains to consider the case $n=2$. Assume that $A\otimes A=V_1\oplus V_2\oplus V_3$ where V_1 is the skew-symmetric part, $V_2\oplus V_3$ the symmetric part and $V_3\cong A$. Let $\pi_i:A\otimes A\to V_i$ $(i=1, 2, 3)$ be the associated projections. If e_1, e_2 is a basis of A then $v_1=e_1\otimes e_2-e_2\otimes e_1$ spans V_1. Define the bilinear form $f:A\otimes A\to \boldsymbol{R}$ by $\pi_1(x\otimes y)=f(x, y)v_1$. Then f is skew-symmetric and $f(\sigma x, \sigma y)=(\det \sigma)f(x, y)$ for all $\sigma\in \mathrm{GL}(A)$. Similarly, define a bilinear form $g:A\otimes A\to \boldsymbol{R}$ by $\pi_2(x\otimes y)=g(x, y)v_2$ where $v_2\in V_2$ is a fixed nonzero tensor. Then g is symmetric and we must have

$$g(\sigma x, \sigma y) = (\det \sigma)^2 g(x, y)$$

for $\sigma\in G_0$ and $x, y\in A$. The factor $(\det \sigma)^2$ is obtained by considering the determinants of the restrictions of $\sigma\otimes\sigma$ to V_i for $i=1, 2, 3$. Both f and g are nondegenerate and we can write $g(x, y)=f(\rho x, y)$ for some fixed $\rho\in\mathrm{GL}(A)$. Then we must have

$$f(\rho\sigma x, \sigma y) = (\det \sigma)^2 f(\rho x, y), \qquad f(\sigma^{-1}\rho\sigma\rho^{-1}x, y) = (\det \sigma)f(x, y).$$

Thus $\pi_1\circ(\sigma^{-1}\rho\sigma\rho^{-1}\otimes I)=(\det\sigma)\pi_1$ and consequently $\sigma^{-1}\rho\sigma\rho^{-1}$ must be a scalar transformation, i.e., $\sigma^{-1}\rho\sigma\rho^{-1}=\pm I$. Since G_0 is connected we have $\rho\sigma=\sigma\rho$ for all $\sigma\in G_0$. It follows from a formula above that $\det\sigma=1$ for $\sigma\in G_0$. Hence g is a G_0-invariant nondegenerate symmetric form. Since G_0 is transitive on $\boldsymbol{P}(A)$ we conclude that g must be definite, say positive definite. By considering A as a Euclidean space with g as an inner product it is clear that $G_0=\mathrm{SO}(2)$.

But now $A\otimes A$ is not a faithful G_0-module because $-I\in\mathrm{SO}(2)$ and we have a contradiction. This completes the proof.

Remark. We could have dismissed the case $n=2$ because it has been been shown by L. Sweet that the two-dimensional homogeneous algebras exist only over GF(2).

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Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada