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FUNCTIONS POSSESSING RESTRICTED MEAN VALUE PROPERTIES[1]

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ABSTRACT. A real-valued function f defined on an open subset of $\mathbf{R}^N$ is said to have the restricted mean value property with respect to balls (spheres) if, for each point x in the set, there exists a ball (sphere) with center x and radius $r(x)$ such that the average value of f over the ball (sphere) is equal to $f(x)$. If f is harmonic then it has the restricted mean value property. Here new conditions for the converse implication are given.

1. **Introduction.** Suppose $D \subseteq \mathbf{R}^N$ is open and $r: D \to \mathbf{R}$ is positive and $r(x) \leqq d(x, D^c)$, the distance from x to the complement of D. Let $B(x, r)$ denote the open ball of radius r about x, and $S(x, r)$ the corresponding sphere. If $f: D \to \mathbf{R}$ satisfies

$$f(x) = \frac{1}{\mu(B(x, r(x)))} \int_{B(x,r(x))} f(y)\mu(dy), \qquad \forall x \in D, \tag{1.1}$$

where μ is Lebesgue measure in $\mathbf{R}^N$, we say that f possesses the restricted mean value property on balls (in D with respect to r). If f satisfies

$$f(x) = \frac{1}{\nu(S(x, r(x)))} \int_{S(x,r(x))} f(y)\nu(dy), \qquad \forall x \in D, \tag{1.2}$$

where $\nu(dy)$ is the element of surface area on $S(x, r(x))$, we say that f possesses the restricted mean value property on spheres. Of course every harmonic f possesses the restricted mean value property on balls and spheres for arbitrary r; we are concerned here with partial converses to this fact.

Several partial converses are known: Courant and Hilbert [4] prove that if D is relatively compact and regular for the Dirichlet problem and f is continuous on $\bar{D}$ and possesses the restricted mean value property on spheres, then f is harmonic. Föllmer [6], using probabilistic techniques, proved a similar result (which he also generalized) requiring r to be Borel measurable, but his only assumption on D is that Brownian paths leave D

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almost surely and f need be continuous only almost everywhere on ∂D and on all of D.

Baxter [2], extending a result of Ackoglu and Sharpe [1], showed that if $\bar{D}$ is a compact C^1 manifold with boundary and r is measurable and satisfies $r(x) \geqq \varepsilon_0 d(x, D^c)$, then any bounded measurable f possessing the restricted mean value property on balls is harmonic.

More recently, Veech [7] and [8] showed that if $N=2$, and in [9], if $N \geqq 1$, D is a relatively compact Lipschitz domain, and r is bounded away from zero on compact subsets of D, then every Lebesgue measurable function whose absolute value is dominated by some harmonic function on D and which possesses the restricted mean value property on balls is harmonic.

In this paper we shall prove the following two theorems:

THEOREM 1. *Suppose f is bounded, D is a proper subset of $\boldsymbol{R}^N$ and r satisfies*

$$\varepsilon d(x, D^c) < r(x) < (1 - \varepsilon) d(x, D^c) \tag{1.3}$$

for some $\varepsilon > 0$. If f possesses the restricted mean value property on balls, then f is harmonic.

THEOREM 2. *Suppose f is bounded, $N \geqq 2$, and r satisfies*

$$|r(x) - r(y)| < (1 - \varepsilon_0)\, |x - y| \tag{1.4}$$

for some $\varepsilon_0 > 0$. If f possesses the restricted mean value property on spheres then f is harmonic.

2. **Preliminary results.** Suppose $(\Omega^0, x^0, \mathscr{F}_t^0, \theta_t^0, P_x^0, x \in \boldsymbol{R}^N)$ is standard N-dimensional Brownian motion. Suppose (ρ_n^1) is a sequence of independent, identically distributed random variables on $(\Omega^1, \mathscr{F}^1, P^1)$ with distribution function G defined by

$$G(x) = \begin{cases} 0, & \text{if } x \leqq 0, \\ x^N, & \text{if } 0 \leqq x \leqq 1, \\ 1, & \text{if } 1 \leqq x. \end{cases}$$

Let $\Omega = \Omega^0 \times \Omega^1$, $P_x = P_x^0 \times P^1$, $\mathscr{F}_t = \mathscr{F}_t^0 \times \mathscr{F}^1$, and define θ_t by

$$\theta_t(\omega^0, \omega^1) = (\theta_t^0(\omega^0), \omega^1).$$

Extend x^0 and ρ_n^1 by $x(t, (\omega^0, \omega^1)) = x^0(t, \omega^0)$ and $\rho_n(\omega^0, \omega^1) = \rho_n^1(\omega^1)$.

Given $r : D \to \boldsymbol{R}$, measurable, with $0 < r(x) < d(x, D^c)$ we define two sequences of stopping times as follows:

$$\tau_0(\omega) = 0,$$
$$\tau_{n+1}(\omega) = \inf\{t : t \geqq \tau_n(\omega), |x(t, \omega) - x(\tau_n(\omega), \omega)| \geqq \rho_{n+1}(\omega) \cdot r(x(\tau_n(\omega), \omega))\}$$

and

$$\sigma_0(\omega) = 0,$$
$$\sigma_{n+1}(\omega) = \inf\{t : t \geqq \sigma_n(\omega), |x(t, \omega) - x(\sigma_n(\omega), \omega)| \geqq r(x(\sigma_n(\omega), \omega))\}.$$

Let $\mathscr{G}_n$ and $\mathscr{H}_n$ denote the minimal σ-fields for the sequences $x(\tau_n)$ and $x(\sigma_n)$ respectively. Then clearly $(x(\tau_n), \mathscr{G}_n)$ is a Markov chain and the conditional distribution of $x(\tau_{n+1})$ given $x(\tau_n)$ is uniform on $B(x(\tau_n), r(x(\tau_n)))$; a similar statement holds for $x(\sigma_{n+1})$ and $S(x(\sigma_n), r(x(\sigma_n)))$.

Define τ by $\tau(\omega)=\inf\{t \geqq 0 : x(t, \omega) \notin D\}$ if this set is nonempty; otherwise set $\tau(\omega)=+\infty$.

LEMMA 1. *If r satisfies* (1.3) *then* $\lim_{n\to\infty} \tau_n(\omega)=\tau(\omega)$ *P_x-a.s. for every $x \in D$. If r satisfies* (1.4) *then* $\lim_{n\to\infty} \sigma_n(\omega)=\tau(\omega)$ *P_x-a.s. for every $x \in D$.*

PROOF. We prove only the first statement: Let $\tau'(\omega)=\lim_{n\to\infty} \tau_n(\omega)$; clearly $\tau' \leqq \tau$ since $\tau_n \nearrow$ and $\tau_n \leqq \tau$. Suppose $\tau'(\omega)<\infty$. Then $x(\tau'(\omega), \omega)= \lim_{n\to\infty} x(\tau_n(\omega), \omega)$. If $x(\tau'(\omega), \omega) \in D$ then $d(x(\tau'(\omega), \omega), D^c)>0$, so by (1.3),

$$\liminf_{n\to\infty} r(x(\tau_n(\omega), \omega)) > 0.$$

But then

$$\limsup_{n\to\infty} \rho_{n+1}(\omega) = \limsup_{n\to\infty} \frac{|x(\tau_{n+1}(\omega), \omega) - x(\tau_n(\omega), \omega)|}{r(x(\tau_n(\omega), \omega))} = 0;$$

this event has probability zero so $\tau'(\omega)<\infty \Rightarrow x(\tau'(\omega), \omega) \notin D \Rightarrow \tau(\omega) \leqq \tau'(\omega)$.

REMARK. We clearly have

$$\bigcup_{i=m}^{\infty} B(x(\tau_i(\omega), \omega), r(x(\tau_i(\omega), \omega)))$$
$$\supseteq \{y : y = x(t, \omega) \text{ for some } t \in [\tau_i(\omega), \tau(\omega))\}.$$

This inclusion remains valid if all τ_i's are replaced by σ_i's.

LEMMA 2. *Suppose f is bounded and measurable on D and satisfies* (1.1) *or* (1.2). *There is a bounded measurable function F on Ω such that $f(x)= E_x(F(\omega))$ $\forall x \in D$.*

PROOF. If f satisfies (1.1) then $f(x(\tau_n(\omega), \omega))$ is a bounded martingale, so by the martingale convergence theorem it converges to some function F P_x-a.s. for every $x \in D$ and the above equation holds. If f satisfies (1.2), replace τ_n by σ_n in the above argument.

For a proof of the following result, see Chung [3]:

LEMMA 3. *Suppose* $(M, \mathfrak{A}, m)$ *is a probability space and* $(\mathfrak{A}_n)$ *is an increasing sequence of* σ*-fields with* $\mathfrak{A}_n \subseteq \mathfrak{A}$. *Let* $\mathfrak{A}_\infty = \bigvee_1^\infty \mathfrak{A}_n$, *and suppose* $A_i \in \mathfrak{A}_\infty$. *Then*

$$\lim_{n\to\infty} P\left[\bigcap_{i=n}^{\infty} A_i \mid \mathfrak{A}_n\right] = 1_{\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} A_i}.$$

LEMMA 4. *Under the hypotheses of Theorem* 2, *there exists a* $\delta > 0$ *such that if* $x_i, A_i, i=1, 2$, *satisfy* $x_i \in D$, A_i *measurable and* $A_i \subseteq D$, $P^2(x_i, A_i) > 1-\delta$ *and* $x_2 \in B(x_1, r(x_1))$ *then* $A_1 \cap A_2 \neq \varnothing$.

PROOF. Define $A(x, r_1, r_2) = \{y : r_1 < |y-x| < r_2\}$, let $\varepsilon' = \varepsilon_0/4$ and let x denote either x_1 or x_2. Suppose $y \in A(x, (1-\varepsilon')r(x), (1+\varepsilon')r(x))$ and a is chosen small enough that $B(y, a) \subseteq A(x, (1-\varepsilon')r(x), (1+\varepsilon')r(x))$. We shall show that

$$P^2(x, B(y, a)) \geqq c(\varepsilon_0)a^N/(r(x))^N. \tag{2.1}$$

To this end, let $C = \{z \in S(x, r(x)) : |r(z) - |z-y|| \leqq a/2\}$. (Note that $|r(z)-|z-y||$ is the distance by which the sphere $S(z, r(z))$ misses the point y.) Simple estimates of the central angle of the part of the sphere $S(z, r(z))$ which is contained in $B(y, a)$ give

$$z \in C \Rightarrow P^1(z, B(y, a)) \geqq c_1(\varepsilon_0)a^{N-1}/(r(x))^{N-1}.$$

Using the inequalities on (and also the continuity of) r, one shows easily that $x + r(x)((y-x)/|y-x|)$ (respectively $x - r(x)((y-x)/|y-x|)$) are not in C because r is too big (small) there, and hence the intersection of C with each great circle on $S(x, r(x))$ through $x + r(x)((y-x)/|y-x|)$ is nonvoid; moreover, the measure of this intersection and the distance of the intersection from $x \pm r(x)((y-x)/|y-x|)$ are easily shown to be bounded below appropriately so that $P^1(x, C) \geqq c_2(\varepsilon_0)a/r(x)$. (2.1) now follows easily. Notice that (2.1) implies that for any measurable

$$E \subseteq A(x, (1-\varepsilon')r(x), (1+\varepsilon')r(x))$$

we have

$$P^2(x, E) \geqq c(\varepsilon_0)\mu(E)/(r(x))^N. \tag{2.2}$$

To complete the proof, set

$$I = A(x_1, (1-\varepsilon')r(x_1), (1+\varepsilon')r(x_1)) \cap A(x_2, (1-\varepsilon')r(x_2), (1+\varepsilon')r(x_2)).$$

An elementary computation (again using the properties of r), using the fact that $x_2 \in B(x_1, r(x_1))$, shows that

$$\mu(I) \geqq k_1(\varepsilon_0) r(x_1)^{N-1} r(x_2) \geqq k(\varepsilon_0) r(x_i)^N, \qquad i = 1, 2. \tag{2.3}$$

We now select δ so that under the hypotheses of the theorem we can conclude $\mu(I \cap A_1^c) + \mu(I \cap A_2^c) < \mu(I)$ so that $A_1 \cap A_2$ must be nonvoid.

From (2.2) we obtain $\mu(I \cap A_i^c) \leqq (r(x_i))^N P^2(x_i, I \cap A_i^c)/c(\varepsilon_0)$, so that by (2.3)

$$\mu(I \cap A_i^c)/\mu(I) \leqq P^2(x_i, I \cap A_i^c)/c(\varepsilon_0)k(\varepsilon_0).$$

Hence if $\delta = c(\varepsilon_0)k(\varepsilon_0)/4$ we obtain

$$\mu(I \cap A_1^c) + \mu(I \cap A_2^c) \leqq \mu(I)/2 < \mu(I)$$

as desired. This completes the proof of Lemma 4.

LEMMA 5. *Suppose r satisfies* (1.3). *Let $P^1(x, A)$ denote the one-step transition probability for the chain $x(\tau_n(\omega), \omega)$. Then there is a $\delta > 0$ such that if x_i, A_i, $i = 1, 2$, satisfy $P^1(x_i, A_i) \geqq 1 - \delta$, $x_i \in D$, $A_i \subseteq D$, $x_2 \in B(x_1, r(x_1))$, then $A_1 \cap A_2 \neq \varnothing$.*

PROOF. Apply simple geometry and use the techniques of the proof of the previous lemma.

3. **Proofs of the theorems.**

PROOF OF THEOREM 1. By a simple modification of an elementary but clever argument of Veech, [7] or [**9**], we may and shall assume that f and r are Borel measurable. Let $x_0 \in D$ and $0 < r_0 < d(x_0, D^c)$. It suffices to show that

$$f(x_0) = \frac{1}{\nu(S(x_0, r_0))} \int_{S(x_0, r_0)} f(y) \nu(dy).$$

Let $y_n(\omega) = x(\tau_n(\omega), \omega)$. By Lemma 2 $\lim_{n\to\infty} f(y_n) = F$ exists P_x-almost surely and $f(x) = E_x(F)$. Let $\eta(\omega) = \inf\{t > 0 : |x(t, \omega) - x_0| \geqq r_0\}$ and θ_η be the corresponding shift operator. Now by the strong Markov property (see Dynkin [**6**, p. 100]):

$$E_{x_0}(F \cdot \theta_\eta \mid \mathscr{F}_\eta) = E_{x_\eta}(F) \quad P_{x_0}\text{-a.s.}, \qquad x_\eta = x(\eta(\omega), \omega).$$

It therefore suffices to show that $F \cdot \theta_\eta = F$ P_{x_0}-a.s., for then

$$\begin{aligned} f(x_0) &= E_{x_0}(F) = E_{x_0}(E_{x_0}(F \mid \mathscr{F}_\eta)) = E_{x_0}(E_{x_0}(F \cdot \theta_\eta \mid \mathscr{F}_\eta)) \\ &= E_{x_0}(E_{x_\eta}(F)) = E_{x_0}(f(x_\eta)) = \frac{1}{\nu(S(x_0, r_0))} \int_{S(x_0, r_0)} f(y) \nu(dy). \end{aligned}$$

Now consider the sequence $z_n(\omega) = y_n(\theta_\eta(\omega))$. Clearly $f(z_n)$ is a martingale which converges to $F \cdot \theta_\eta(\omega)$ P_{x_0}-a.s. Thus we need only show that $\lim_{n\to\infty} f(z_n(\omega)) = \lim_{n\to\infty} f(y_n(\omega))$.

From the definition of the process z_n, an argument similar to the proof of Lemma 1, and the remark following Lemma 1 we see that: For each m there is an $n_0(\omega)$ such that $n \geqq n_0(\omega) \Rightarrow z_n(\omega) \in \bigcup_{i=m}^{\infty} B(y_i(\omega), r(y_i(\omega)))$.

Suppose now that $\varepsilon > 0$ and $K > 0$ are prescribed. Let

$$A_i^\varepsilon = \{\omega : |f(y_{i+1}(\omega)) - f(y_i(\omega))| < \varepsilon/2\},$$
$$B_i^\varepsilon = \{\omega : |f(z_{i+1}(\omega)) - f(z_i(\omega))| < \varepsilon/2\}.$$

Because of the convergence mentioned above, we have

$$P_{x_0}\left(\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} A_i^\varepsilon\right) = 1, \qquad P_{x_0}\left(\bigcup_{m=1}^{\infty} \bigcap_{i=m}^{\infty} B_i^\varepsilon\right) = 1,$$

so by Lemma 3 and the strong Markov property we have

$$\lim_{m\to\infty} P_{x_0}\left(\bigcap_{i=m}^{\infty} A_i^\varepsilon \mid y_m\right) = 1, \qquad \lim_{m\to\infty} P_{x_0}\left(\bigcap_{i=m}^{\infty} B_i^\varepsilon \mid z_m\right) = 1 \quad P_{x_0}\text{-a.s.},$$

and hence

$$\lim_{m\to\infty} P_{x_0}(A_m^\varepsilon \mid y_m) = 1, \qquad \lim_{m\to\infty} P_{x_0}(B_m^\varepsilon \mid z_m) = 1 \quad P_{x_0}\text{-a.s.}$$

Now let $C^\varepsilon(x) = \{y \in D : |f(y) - f(x)| < \varepsilon/2\}$. It is easy to see that

$$P_{x_0}(A_m^\varepsilon \mid y_m) = P(y_m, C^\varepsilon(y_m)), \qquad P_{x_0}(B_m^\varepsilon \mid z_m) = P(z_m, C^\varepsilon(z_m))$$

and so the right-hand sides converge to one almost surely. Let δ be as in Lemma 5, and choose $m_0 \geqq K$ so large that $m \geqq m_0 \Rightarrow p(y_m, C^\varepsilon(y_m)) \geqq 1 - \delta$. Choose $n_0(\omega)$ so large that $n \geqq n_0 \Rightarrow$

$$z_n \in \bigcup_{i=m_0}^{\infty} B(y_m, r(y_m)), \qquad p(z_n, C^\varepsilon(t_n)) \geqq 1 - \delta, \qquad n_0 \geqq K.$$

Then let $m_1 \geqq m_0$ satisfy $z_{n_0} \in B(y_{m_1}, r(y_{m_1}))$. Applying Lemma 5 to the points $x_1 = y_m$, $x_2 = z_{n_0}$, $A_1 = C^\varepsilon(y_m)$, $A_2 = C^\varepsilon(z_{n_0})$ we see that $C^\varepsilon(y_m) \cap C^\varepsilon(z_{n_0}) \neq \varnothing$, so there is an $x \in D$ with

$$|f(x) - f(y_m)| < \varepsilon/2 \quad \text{and} \quad |f(x) - f(z_{n_0})| < \varepsilon/2$$

and thus $|f(y_m) = f(z_{n_0})| < \varepsilon$. Because K was arbitrary, we must have

$$\left|\lim_{n\to\infty} f(y_n) - \lim_{n\to\infty} f(z_n)\right| \leqq \varepsilon$$

whenever these limits exist, P_{x_0}-a.s. Hence $F = F \cdot \theta_\eta$ P_{x_0}-a.s. and the proof is completed.

PROOF OF THEOREM 2. Theorem 2 is proved in the same manner as Theorem 1, using instead the two-step transition probability for the

Markov chains y'_n and z'_n given by

$$y'_n(\omega) = x(\sigma_n(\omega), \omega), \qquad z'_n(\omega) = y'_\eta(\theta_n(\omega)).$$

4. **Examples.** The first example shows that some restrictions on r are necessary. Let D be the unit disc in $\boldsymbol{R}^2$, i.e., $\{(x, y): x^2+y^2<1\}$ and define

$$\begin{aligned} f(x, y) &= 1, \quad \text{if } y > 0, \\ &= 0, \quad \text{if } y = 0, \\ &= -1, \quad \text{if } y < 0. \end{aligned}$$

Let $r(x, y)=\min(|y|, 1-\|(x, y)\|)$ for $y\neq 0$ and $1-|x|$ for $y=0$. Then f satisfies (1.1), is bounded and measurable, but not harmonic.

The second example shows that some restrictions on the growth of f are necessary. Let $D=(0, 1)$ in $\boldsymbol{R}^1$ and define

$$f(x) = \sum_{n=4}^{\infty} \varphi(n) 1_{[(n-1)/n, n/(n+1))}(x)$$

where $\varphi(k)=0$, if $k=1, 2, 3$, and $\varphi(k)=(-1)^k k^2(k+1)$, for $k>3$.

Notice that

$$\int_0^{n/(n+1)} f(x)\, dx = \sum_{k=4}^{n} (-1)^k k^2(k+1) \frac{1}{k(k+1)} = \sum_{k=4}^{n} (-1)^k k.$$

Then if $g(x)=\int_0^x f(y)\, dy$, $\limsup_{x\to 1} g(x)=+\infty$ and $\liminf_{x\to 1} g(x)=-\infty$ and g is continuous on $[0, 1)$. Moreover for any $x \in (1/2, 1)$ the function

$$h_x(r) = \frac{g(x+r)-g(x-r)}{2r} \quad \text{for } 0 < r < 1-x$$

is continuous and also has lim sup and lim inf of $+\infty$ and $-\infty$ respectively as $r\to 1-x$. Thus there are values of r arbitrarily near $1-x$ for which

$$f(x) = h_x(r). \tag{4.1}$$

Clearly r can be chosen a measurable function of x to satisfy (4.1). For $0<x\leqq\frac{1}{2}$ one can define $r(x)\equiv\frac{1}{4}$. Thus f and r satisfy all of the hypotheses of our theorem except that of boundedness, and f is not harmonic. (A similar example can be constructed in which f is bounded and $D=(-\infty, \infty)$ by setting

$$f(x) = \sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{1}{n}\right) 1_{[a_n, a_{n+1})}(|x|)$$

where (a_n) increases sufficiently rapidly.)

Finally, we remark that any bounded measurable periodic function on $\mathbf{R}^1$ satisfies the hypotheses for Theorem 2 with $r(x)$ equal to the period except that $N=1$. For further examples related to these concepts, see Courant and Hilbert [**4**].

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