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REPRESENTATION OF A-CONVEX ALGEBRAS

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ABSTRACT. Algebraic properties of A-convex algebras are developed via a functor to locally m-convex algebras. The Gel'fand-Mazur theorem holds for A-convex algebras, and this fact allows a Gel'fand-type representation theorem for a subclass of uniformly A-convex algebras. Connections to existing functional representation theory are also obtained.

1. **Introduction.** The objects of this note are to give some algebraic properties of A-convex algebras, to note the fact that the Gel'fand-Mazur theorem holds for this class of algebras and to develop a Gel'fand-type theorem appropriate for certain A-convex algebras. The main technique for obtaining algebraic properties is to use a functor from the category of A-convex algebras to the category of locally m-convex algebras and then use the existing theory of m-convex algebras (see, for example, **[5]**). Various relations are developed along these lines.

The fact that the Gel'fand-Mazur theorem holds for A-convex algebras means that for commutative algebras there is a one-to-one correspondence between continuous nonzero multiplicative linear functionals and the closed regular maximal ideals. However, the generalizations of the Gel'fand representation theory on this carrier space (e.g. **[6]**) are all in terms of a locally m-convex algebra. We develop a representation theorem for a subclass of A-convex algebras (uniformly A-convex) which gives the m-convex results as corollaries. A type of strict topology is used on the space of continuous functions in which the algebra is embedded.

Basic results on A-convex algebras are found in **[1]**, **[2]** and **[3]**. In order to make this note relatively self-contained we briefly repeat pertinent definitions. A *locally convex algebra* is an algebra over $\boldsymbol{R}$ or $\boldsymbol{C}$ with a locally convex topology for which multiplication is separately continuous. An *A-convex seminorm* on an algebra E is a seminorm p such that for $x \in E$ there are constants M_x and N_x such that $p(xy) \leqq M_x p(y)$ and $p(yx) \leqq N_x p(y)$ for all $y \in E$. An *A-convex algebra* {*locally m-convex algebra*}

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is a locally convex algebra whose topology is defined via a family of A-convex seminorms {multiples of submultiplicative seminorms}. A basic reference to locally m-convex algebras is [**5**]. Examples of A-convex algebras (see [**2**]) include: (1) all locally m-convex algebras; (2) the space $(C_b(x), \beta)$ of bounded $\boldsymbol{C}$-valued functions on a locally compact Hausdorff space with the strict topology; and (3) multiplier algebras.

We will denote by $x \circ y$ the operation on E defined by $x \circ y = x + y - xy$. If $x \circ y = y \circ x = 0$ then x is called the *quasi-inverse* of y.

2. **Algebraic properties.** The Gel'fand-Mazur theorem forms the cornerstone of the basic representation theorem for Banach algebras [**7**]. Indeed, one of the important features of locally m-convex algebras is the fact that this theorem holds in this class of algebras [**5**]. Several authors (e.g. [**5**], [**4**], [**6**]) have exploited this type of representation theory, some obtaining results by assuming a sufficiently large carrier space (which the Gel'fand-Mazur theorem assures). We note here the fact that the Gel'fand-Mazur theorem holds for the larger class of A-convex algebras. Then we note several algebraic properties of these algebras, reducing to the m-convex and/or Banach case such questions as existence of identity, inverses and quasi-inverses.

THEOREM 2.1 (GEL'FAND-MAZUR). *Every A-convex division algebra over* $\boldsymbol{C}$ *is isomorphic to the complex field* $\boldsymbol{C}$.

PROOF. This follows immediately from a remark of Williamson [**13**, p. 730].

It is of some interest to note the departure of the m-convex theory at this point. Theorem 2.1 for m-convex algebras is a direct transplant from the Banach algebra theorem. One simply notes that quasi-inversion is continuous for m-convexity. But a result of Turpin [**8**] shows that quasi-inversion is never continuous in an A-convex algebra which is not m-convex. Thus $(C_b(\boldsymbol{R}), \beta)$ gives an easy example of an algebra with quasi-inversion not continuous.

In view of the representation theorem promised, we need the following simple consequences of Theorem 2.1:

THEOREM 2.2. *Let E be a commutative complex A-convex algebra.*

(i) *If M is a closed regular maximal ideal in E then E/M is isomorphic to* $\boldsymbol{C}$;

(ii) *There is one-to-one correspondence between the set* $\mathfrak{M}$ *of nonzero continuous multiplicative linear functionals and the closed maximal regular ideals of E in which a functional is associated with its kernel.*

One of the main features of complete locally m-convex algebras is the theorem of Michael [**5**, Theorem 5.2] which reduces to Banach algebras the questions of existence of identity, inverses and quasi-inverses. Let **M** denote the category of locally m-convex algebras with continuous algebraic homomorphisms and, similarly, **A** the category of A-convex algebras. Warner [**11**] showed that $A_1 = \bigcup \{A^n : n = 1, 2, \cdots\}$ is the smallest idempotent subset of an algebra E which contains A. Let A^* denote the absolutely convex hull of A_1 for a given set A.

PROPOSITION 2.3. *Let (E, τ) be an A-convex algebra with $\mathcal{N}$ the neighborhood filter at zero. Then the filter $m(\mathcal{N})$, generated by all sets V^*, V in $\mathcal{N}$, gives a locally m-convex topology on E which is the finest locally m-convex topology on E coarser than τ. The correspondence thus described gives an idempotent functor $m: \mathbf{A} \to \mathbf{M}$.*

PROOF. The proof of the first part is given in [**2**]; the second part follows from the fact that if f is an algebraic homomorphism then $[f(A^*)] \subset [f(A)]^*$ and similar straightforward calculations.

By using the functor m of Proposition 2.3 the problems of determining algebraic properties of an A-convex algebra are reduced to the locally m-convex case. The filter $m(\mathcal{N})$ is defined for any locally convex algebra so m can be considered as a functor on all such algebras.

PROPOSITION 2.4. *Let E be an A-convex algebra, $m(E)$ a complete Hausdorff algebra. Then the existence of identity, inverses and quasi-inverses is equivalent to their existence in a set of Banach algebras.*

PROOF. Clearly, existence of identity, inverses and quasi-inverses is independent of the topology on E. Hence Michael [**5**, Theorem 5.2] gives the result.

EXAMPLE 2.5. The functor m associates $(C_b(X), \kappa)$ with the space $(C_b(X), \beta)$. This is shown in [**2**] so that there are no locally m-convex topologies on $C_b(X)$ between the compact-open and the strict.

THEOREM 2.6. *For any locally convex algebra E, the set of continuous multiplicative linear functionals is the same for both E and $m(E)$.*

PROOF. A linear functional is continuous if and only if it is bounded on a neighborhood of zero. Thus, if m is a multiplicative linear functional which is bounded on a neighborhood V in E then for a suitable multiple W of V, $m(W) \leq 1$. Since m is multiplicative $m(W^*) \leq 1$ so m is continuous in $m(E)$. Since the topology of $m(E)$ is coarser than the topology of E, the result follows. We now consider the question of when $m(E)$ is Hausdorff and/or complete.

DEFINITION 2.7. For any locally convex algebra E with local neighborhood filter $\mathcal{N}$, an element $x \in E$ is *topologically nilpotent* if x is an element of $V_0 = \bigcap \{V^* : V \in \mathcal{N}\}$. Denote the set of topological nilpotents by $\mathcal{E}$.

LEMMA 2.8. *Let E be a locally convex algebra. The subspace is precisely the closure of* (0) *in $m(E)$ and contains all $x \in E$ such that x^n converges to* 0 *as $n \to \infty$.*

PROOF. Immediate from the definition of m and the fact that the closure of (0) is the intersection of all local neighborhoods.

THEOREM 2.9. *Let E be a locally convex algebra. Then $m(E)$ is Hausdorff if and only if $\mathscr{E} = (0)$. Further, $\mathscr{E}$ is closed with respect to quasi-inversion.*

PROOF. A locally convex space is closed if and only if $\mathrm{cl}(0) = (0)$ so Lemma 2.8 gives the first part. The second part is an easy consequence of the definition.

The result of Theorem 2.9 is that quasi-inverses may be investigated by checking the space $\mathscr{E}$ together with the Hausdorff m-convex algebra $m(E)/\mathscr{E}$.

There is no relation between completeness of E and completeness of $m(E)$. Using Example 2.5, $(C_b(\mathbf{R}), \beta)$ is complete but $(C_b(\mathbf{R}), \kappa)$ is not complete. Wang **[10]** gives an example where both β and κ are complete. The A-normed algebra given in **[1]** has the property that neither E nor $m(E)$ (the compact-open on $C_b(0, 1)$) is complete. However, for function spaces the determination of completeness is often easily checked.

Many of the m-convex consequences of Theorem 5.2 of **[5]** have immediate analogues for the A-convex setting which are not included here.

3. **Representation.** Morris and Wulbert **[6]** gave a general situation for representation of a commutative locally convex algebra as a subspace of some $(C(X), \kappa)$ where X is at least completely regular and κ denotes the compact-open topology. Since κ always gives a locally m-convex topology, there is no hope of a satisfactory representation theorem in the A-convex case. For a subclass of A-convex algebras, we provide a suitable representation theorem here. As a consequence, the relations expressed by A. Mallios **[4]** in terms of "m-barrelled" can be viewed in a different way. The fact that m-barrelled A-convex algebras (see **[1]**) are m-convex provides the bridge between our theorem and some of the results of **[4]** and **[6]**.

Let E be a commutative A-convex algebra and $\mathfrak{M}$ the space of nonzero continuous multiplicative linear functionals topologized via the $\sigma(E', E)$ topology. Theorem 2.2 gives the usual relationship with maximal ideals. Let G denote the Gel'fand map of $E \to C(\mathfrak{M})$ given by $G(x)(m) = m(x) = \hat{x}(m)$. Then G gives an algebraic isomorphism if and only if E is semisimple. For A-convex algebras, semisimple is equivalent to "strongly semisimple" as defined in **[6]**.

DEFINITION 3.1. A locally convex algebra E is *uniformly A-convex* if there is some defining family of seminorms $\{p_\alpha : \alpha \in A\}$ with the property that for each x in E there are constants M_x and N_x such that

$$p_\alpha(xy) \leqq M_x p_\alpha(y) \quad \text{and} \quad p_\alpha(yx) \leqq N_x p_\alpha(y)$$

for all $\alpha \in A$ and for all $y \in E$. (Such a family will be called a *uniform family*.)

Clearly, uniformly A-convex algebras are A-convex. Examples include $(C_b(X), \beta)$, $(C_b(X), \kappa)$ and A-normed [3] algebras. The space $(C(X), \kappa)$ is an m-convex algebra which is not, in general, uniformly A-convex (this last fact can be seen from the sequel).

Let E be a commutative A-convex algebra with identity e. Then a uniform defining family $\{p_\alpha : \alpha \in A\}$ can be chosen such that $p_\alpha(e)=1$ for all $\alpha \in A$. If E does not have an identity, a uniform family can be chosen with this property on E^+, the A-convex algebra obtained by adjoining an identity. We assume such a family is always chosen in the following results. Define

$$(*) \qquad \|x\| = \inf\{M_x : p_\alpha(xy) \leqq M_x p_\alpha(y), \text{ for all } \alpha \in A, y \in E\}.$$

Then it is easy to verify that

$$(**) \qquad \|x\| = \sup[\sup\{p_\alpha(xy) : p_\alpha(y) \leqq 1\} : \alpha \in A].$$

LEMMA 3.2. *The norm defined by* (*) *is a submultiplicative norm on E for which $p_\alpha(x) \leqq \|x\|$ for all $\alpha \in A$, $x \in E$.*

PROOF. This result is obtained by using (**) and properties of suprema. If E has an identity, $p_\alpha(x)=p_\alpha(xe) \leqq M_x p_\alpha(e)=M_x$ for all α, and $\|x\|$ is the inf of all such constants M_x so $p_\alpha(x) \leqq \|x\|$. If E does not have an identity, one can be adjoined and the above argument can be applied.

Let E_n denote the normed space $(E, \| \; \|)$ and $\mathcal{M}_n$ the carrier space of E_n. The respective dual spaces are related by $E_n \supset E'$ and the structure spaces by $\mathfrak{M}_n \supset \mathfrak{M}$. If E_n is complete we have

THEOREM 3.3. *The structure space $\mathfrak{M}$ is locally compact and the Gel'fand transform of x in E is the restriction of the Gel'fand transform of x in E_n. $\mathfrak{M}$ is Hausdorff if and only if E is semisimple.*

PROOF. Since $\mathfrak{M}_n$ is locally compact [7] and $\mathfrak{M}$ is a closed subset, $\mathfrak{M}$ is locally compact. The remainder of the theorem is clear.

For each $\alpha \in A$, define the extended real number

$$t_\alpha(m) = \sup\{|\hat{x}(m)| : p_\alpha(x) \leqq 1\}.$$

Then define $\phi_\alpha : \mathfrak{M} \to R_+$ by $\phi_\alpha(m)=0$, if $t_\alpha(m)=+\infty$, and $\phi_\alpha(m)=1/t_\alpha(m)$ otherwise.

Note that if E has an identity then $0 \leqq \phi_\alpha(m) \leqq 1$ since $\hat{e}(m)=1$ gives $t_\alpha(m) \geqq 1$. Let H be the subspace of $C(\mathfrak{M})$ defined by

$$H = \{f \in C(\mathfrak{M}) : \phi_\alpha f \text{ is bounded for all } \alpha \in A\}.$$

Clearly $C_b(\mathfrak{M}) \subset H$.

THEOREM 3.4. *Let E be a commutative uniformly A-convex algebra with Gel'fand map G. Then $G(E) \subset C_b(\mathfrak{M}) \subset H$.*

PROOF. Since $|\hat{x}(m)| = |m(x)| \leqq \|x\|$, $G(E) \subseteq C_b(\mathfrak{M})$.

Let β denote the topology on H defined by the family of seminorms $\{\hat{p}_\alpha : \alpha \in A\}$ where

$$\hat{p}_\alpha(f) = \sup\{|f(m)\phi_\alpha(m)| : m \in \mathfrak{M}\}, \qquad f \in H.$$

This gives a type of strict topology on H replacing the compact-open in [**4**].

THEOREM 3.5. *Let E be uniformly A-convex and G the Gel'fand transform. Then G is continuous from E into (H, β). If E is semisimple then E is represented via the continuous isomorphism G.*

PROOF. If $p_\alpha(x) \neq 0$, let $y = [p_\alpha(x)]^{-1}x$. *Then* $p_\alpha(y)=1$ *and* $m(x) = p_\alpha(x)m(y)$. *From the definition of* $\hat{p}_\alpha(\hat{x})$, $\hat{p}_\alpha(\hat{x}) \leqq p_\alpha(x)\sup\{|m(y)|/t_\alpha(m)\} \leqq p_\alpha(x)$. If $p_\alpha(x)=0$ then $p_\alpha(kx)=0$ for every natural number k. Hence $t_\alpha(m) = +\infty$, $\phi_\alpha(m)=0$ and $\hat{p}_\alpha(\hat{x}) = 0 \leqq p_\alpha(x)$. Thus, $\hat{p}_\alpha(\hat{x}) \leqq p_\alpha(x)$ for all $x \in E$ so G is continuous.

REMARK 3.6. The seminorms p can often be defined in more general cases, and Theorem 3.5 still holds with the same proof. We note also that if all the ϕ_α vanish at ∞ and generate $C_0(\mathfrak{M})$ then we get the strict topology. If there is but one ϕ_α which is bounded away from zero we get the normed case. Thus, the representation theorem given here is general enough to include previous results.

DEFINITION 3.7. A commutative uniformly A-convex algebra is *saturated* if for a uniform defining family $\{p_\alpha : \alpha \in A\}$ the following property holds for each p_α: for each x in E such that $p_\alpha(x)=1$ there exists $m_0 \in \mathfrak{M}$ such that $m_0(x) = \sup\{|\hat{y}(m)| : p_\alpha(y) \leqq 1\}$ for some $m \in \mathfrak{M}$. (Then $\hat{p}_\alpha(\hat{x}) = 1 = p_\alpha(x)$.)

THEOREM 3.8. *If E is a commutative saturated uniformly A-convex algebra then G is an open map. Hence if E is semisimple, G is a topological and algebraic isomorphism.*

PROOF. If E is saturated then $\hat{p}_\alpha(\hat{x}) = p_\alpha(x)$ for all $x \in E$ and $\alpha \in A$. Hence if E is semisimple G is a bijection onto $G(E)$ with G and G^{-1} continuous.

REMARK 3.9. If G is an algebraic and topological isomorphism and E is barrelled (or i-barrelled [11] and [4]) then E is locally m-convex. This gives a connection to the results in [4] and those of this section. In any case, with the conditions of Theorem 3.5 we may apply the functor m of §2 to obtain a representation in a locally m-convex algebra. Since application of m gives a coarser topology, G is still continuous. Of course, if $G(E)$ is $(C_b(X), \beta)$ then this gives $(C_b(X), \kappa)$.

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