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COMPACT SUBSETS OF R^n AND DIMENSION OF THEIR PROJECTIONS

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ABSTRACT. In this paper it is proved that a k-dimensional closed subset $X\subset R^n$ admits a projection p into one of the coordinate k-planes such that $\dim p(X)=k$.

The purpose of this note is to prove the following theorem:

THEOREM. *Let $X\subset R^n$ be a k-dimensional compact subset of R^n, $1\leqq k\leqq n$. Then there exist k different factors $R_{i_1}=R,\cdots,R_{i_k}=R$ of R^n, $1\leqq i_1<\cdots<i_k\leqq n$, such that* $\dim p_{i_1\cdots i_k}(X)=k$, *where $p_{i_1\cdots i_k}$ is the projection $R^n\to R_{i_1}\times\cdots\times R_{i_k}$.*

The question of whether the above statement is true was raised by J. D. Lawson in connection with a problem concerning n-dimensional topological semilattices on a Peano continuum. I am indebted to J. Nagata for bringing it to my attention.

It is well known that a subset $Y\subset R^k$ is k-dimensional if and only if it has nonempty interior, Int $Y\neq\varnothing$ (see, e.g., [**1**, Theorem IV.3, p. 44]). Consequently, the theorem can be given the following equivalent form:

THEOREM. *Let $X\subset R^n$ be a k-dimensional compact subset of R^n, $1\leqq k\leqq n$. Then there exist $1\leqq i_1<\cdots<i_k\leqq n$ such that* $\operatorname{Int} p_{i_1\cdots i_k}(X)\neq\varnothing$ *in $R_{i_1}\times\cdots\times R_{i_k}$.*

We prove the theorem by induction on n using this second form. If $n=1$, then $k=1$ and $p_1\colon R\to R$ is the identity so that $p_1(X)=X$. However, $X\subset R$ must contain a nonempty open set for otherwise $R\backslash X$ would be dense in X, which would imply $\dim X\leqq 0$ (inductive dimension), contradicting the assumption $\dim X=1$.

We now assume that the theorem holds for positive integers $\leqq n-1$, $n\geqq 2$, and we prove it for n. Let $X\subset R^n$, $\dim X=k$, $k\leqq n$. Consider any of

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the n factors of R^n, say $R_1=R$, so that $R^n=R_1\times R^{n-1}$. Let $S_1\subset R_1$ be the set of all points $\xi_1\in R_1$ such that

$$\dim(X\cap(\xi_1\times R^{n-1}))\leqq k-2. \tag{1}$$

Furthermore, for any $k-1$ different integers $2\leqq i_1<\cdots<i_{k-1}\leqq n$, consider all balls $B_{i_1\cdots i_{k-1}}(q,\varepsilon)\subset R_{i_1}\times\cdots\times R_{i_{k-1}}$ with rational radius $\varepsilon>0$ and center $q=(q_{i_1},\cdots,q_{i_{k-1}})$ all of whose coordinates are rational. Let $S_{i_1\cdots i_{k-1}}(q,\varepsilon)$ be the set of all points $\xi_1\in R_1$ such that

$$\xi_1\times B_{i_1\cdots i_{k-1}}(q,\varepsilon)\subset p_{1i_1\cdots i_{k-1}}(X\cap(\xi_1\times R^{n-1})). \tag{2}$$

We shall first show that

$$R_1\backslash S_1\subset\bigcup S_{i_1\cdots i_{k-1}}(q,\varepsilon), \tag{3}$$

where the union is taken over all sequences $2\leqq i_1<\cdots<i_{k-1}\leqq n$ and over all rational (q,ε) and thus has countably many terms. Indeed, if $\xi_1\in R_1\backslash S_1$, then $\dim(X\cap(\xi_1\times R^{n-1}))=l\leqq n-1$ is $k-1$ or k. By the induction hypothesis, there is a sequence $2\leqq i_1<\cdots<i_l\leqq n-1$ such that the set $p_{1i_1\cdots i_l}(X\cap(\xi_1\times R^{n-1}))$ contains a nonempty open subset of $\xi_1\times R_{i_1}\times\cdots\times R_{i_l}$ and a fortiori contains a ball $\xi_1\times B_{i_1\cdots i_l}(q,\varepsilon)$ with (q,ε) rational. If $l=k-1$, this yields (2) and thus $\xi_1\in S_{i_1\cdots i_{k-1}}(q,\varepsilon)$. If $l=k$, we consider the projection

$$p:R_1\times R_{i_1}\times\cdots\times R_{i_{k-1}}\times R_{i_k}\to R_1\times R_{i_1}\times\cdots\times R_{i_{k-1}}$$

and note that

$$p_{1i_1\cdots i_{k-1}}=pp_{1i_1\cdots i_k},$$
$$p(\xi_1\times B_{i_1\cdots i_k}(q,\varepsilon))=\xi_1\times B_{i_1\cdots i_{k-1}}(p(q),\varepsilon)$$

and $(p(q),\varepsilon)$ is rational. Consequently, $p_{1i_1\cdots i_{k-1}}(X\cap(\xi_1\times R^{n-1}))$ contains $\xi_1\times B_{i_1\cdots i_{k-1}}(p(q),\varepsilon)$ and therefore $\xi_1\in S_{i_1\cdots i_{k-1}}(p(q),\varepsilon)$. Formula (3) is thus established.

If a given set $S_{i_1\cdots i_{k-1}}(q,\varepsilon)$ intersects a nondegenerate interval $I\subset R_1$ in a set D which is dense in I, then by (2),

$$D\times B_{i_1\cdots i_{k-1}}(q,\varepsilon)\subset p_{1i_1\cdots i_{k-1}}(X). \tag{4}$$

Since $p_{1i_1\cdots i_{k-1}}(X)$ is compact and $\bar{D}=I$, (4) implies

$$I\times B_{i_1\cdots i_{k-1}}(q,\varepsilon)\subset p_{1i_1\cdots i_{k-1}}(X). \tag{5}$$

Consequently, in $R_1\times R_{i_1}\times\cdots\times R_{i_{k-1}}$

$$\operatorname{Int}p_{1i_1\cdots i_{k-1}}(X)\neq\varnothing. \tag{6}$$

We have thus either a projection

$$p_{1i_1\cdots i_{k-1}}: R^n \to R_1 \times R_{i_1} \times \cdots \times R_{i_{k-1}}, \qquad 1 < i_1 < \cdots < i_{k-1} \leqq n,$$

satisfying (6), or every set $S_{i_1\cdots i_{k-1}}(q, \varepsilon)$ is nowhere dense in R_1. However, in the latter case, by Baire's theorem, $R_1 \backslash \bigcup S_{i_1\cdots i_{k-1}}(q, \varepsilon)$ must be dense in R_1. It then follows from (3) that S_1 too is a dense subset of R_1.

The same argument applies to any other $j \in \{1, \cdots, n\}$ and we conclude that either there is a projection $p_{i_1\cdots i_k}: R^n \to R_{i_1} \times \cdots \times R_{i_k}$, $1 \leqq i_1 < \cdots < i_k \leqq n$, such that $j \in \{i_1, \cdots, i_k\}$ and $\operatorname{Int} p_{i_1\cdots i_k}(X) \neq \varnothing$ in $R_{i_1} \times \cdots \times R_{i_k}$ or the set $S_j \subset R_j$ of all $\xi_j \in R_j$ satisfying

$$(7) \quad \dim(X \cap (R_1 \times \cdots \times R_{j-1} \times \xi_j \times R_{j+1} \times \cdots \times R_n) \leqq k - 2$$

is dense in R_j.

However, S_j cannot be dense in R_j for all $j \in \{1, \cdots, n\}$. Indeed, that would imply that every point $x \in X$ admits arbitrarily small neighborhoods $U = (\alpha_1, \beta_1) \times \cdots \times (\alpha_n, \beta_n) \subset R^n$, where $\alpha_j, \beta_j \in S_j$ for all j. Since, by (7), the boundary of U meets X in a set of dimension $\leqq k-2$, we would have $\dim X \leqq k-1$, which contradicts the assumption. This completes the proof of the theorem.

REMARK 1. For $k=n$ we have here an alternate proof for the fact that an n-dimensional compact subset $X \subset R^n$ has a nonempty interior.

REMARK 2. A compact subset $X \subset R^n$ need not be of dimension $\dim X \geqq k$ if it admits a projection $p_{i_1\cdots i_k}: R^n \to R_{i_1} \times \cdots \times R_{i_k}$ with $\dim p_{i_1\cdots i_k}(X) = k$. E.g., let $I = [0, 1]$ and let $f: I \to I^2$ be a continuous surjection (I^2 is a Peano continuum). Then $X = \{t \times f(t) \mid t \in I\} \subset R^3$ is an arc and $\dim p_{23}(X) = 2$.

REMARK 3. The conclusion of the theorem remains true if one weakens the assumptions to X being a closed k-dimensional subset of R^n. Indeed, every closed X is the union of a sequence of compact subsets $X_i \subset R^n$, $i = 1, 2, \cdots$. Since $k = \dim X = \max\{\dim X_i \mid i = 1, 2, \cdots\}$, there is an i such that $\dim X_i = k$ and the conclusion follows from the one in the compact case.

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