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FIELDS OF CONSTANTS OF INFINITE HIGHER DERIVATIONS

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ABSTRACT. Let K be a field of characteristic $p\neq 0$, and let P be its maximal perfect subfield. Let h be a subfield of K containing P such that K is separable over h. We prove: Every regular subfield of K containing h is the field of constants of a set of higher derivations on K if and only if (1) the transcendence degree of K over h is finite, and (2) K has a separating transcendency basis over h. This result leads to a generalization of the Galois theory developed in [4].

I. **Introduction.** Let K be a field of characteristic $p\neq 0$, and let P be its maximal perfect subfield. If h is the field of constants of a set of higher derivations on K, then h is a regular subfield of K containing P. This paper is concerned with determining when every regular subfield of K containing h (and hence P) is the field of constants of a set of higher derivations on K. Necessary and sufficient conditions are shown to be (1) the transcendence degree of K over h is finite, and (2) K has a separating transcendency basis over h. This is Corollary (4.2). This result leads to an immediate extension of the Galois theory developed in [**4**]. In part, we can restate the main result of [**4**] as follows: Assume K has a finite separating transcendency basis over a subfield h containing P. Then there exists a one-to-one correspondence between regular subfields of K containing h and Galois subgroups of $H_h^\infty(K)$. (The characterization of Galois subgroups remains the same.) Moreover, (4.2) shows this to be the most general condition on K relative to h under which all regular subfields of K containing h will be fields of constants of groups of higher derivations on K.

II. **Definitions and preliminary results.** Throughout this paper, K will be a field of characteristic $p\neq 0$. A higher derivation on K is a sequence $d=\{d_i|0\leqq i<\infty\}$ of additive maps of K into K such that

$$d_r(ab) = \sum \{d_i(a)d_j(b) \mid i+j=r\}$$

and d_0 is the identity map. The set $H^\infty(K)$ of all higher derivations on K

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is a group with respect to the composition $d \circ e = f$ where

$$f_i = \sum \{d_m e_n \mid m + n = i\}$$

[1, Theorem 1, p. 33]. Note that the first nonzero map (of subscript>0) is a derivation. The field of constants of a subset $G \subseteq H^\infty(K)$ is $\{a \in K \mid d_i(a) = 0, i > 0, (d_i) \in G\}$. $H_h^\infty(K)$ will denote the group of all higher derivations on K whose field of constants contains the subfield h.

(2.1) [2, Theorem 1]. Let B be a p-basis for K and let $f: A \times B \to K$ be an arbitrary function. There is a unique $(d_i) \in H^\infty(K)$ such that for each $b \in B$ and $i \in Z$, $d_i(b) = f(i, b)$.

(2.2) [3, p. 436]. Let $(d_i) \in H^\infty(K)$ and $a \in K$. Then $d_{ip}(a^p) = (d_i(a))^p$ and if p and j are relatively prime, $d_j(a^p) = 0$.

A field K is a regular extension of a subfield h if K/h is separable and h is algebraically closed in K.

(2.3) [4, Theorem 2.3]. Let h be the field of constants of a set of higher derivations on K. Then K is a regular extension of h.

(2.4) Lemma. *The field of constants of $H^\infty(K)$ is P, the maximal perfect subfield of K.*

Proof. Let $\alpha \in K|P$. If $\{\alpha, \alpha^{p^{-1}}, \alpha^{p^{-2}}, \cdots\} \subseteq K$, then $P(\alpha, \alpha^{p^{-1}}, \alpha^{p^{-2}}, \cdots)$ would also be perfect, contrary to the assumption that P is maximal. Thus there exists $n \geqq 0$ such that $\alpha^{p^{-n}} \in K \backslash K^p$. Let $\{\alpha^{p^{-n}}\} \cup T$ be a p-basis for K, and define $d = \{d_i\}$ by $d_1(\alpha^{p^{-n}}) = 1$, $d_1(t) = 0 \ \forall t \in T$, $d_i(x) = 0$, $x \in \{\alpha^{p^{-n}}\} \cup T$, $1 < i < \infty$. Then

$$d_{p^n}(\alpha) = d_{p^n}((\alpha^{p^{-n}})^{p^n}) = (d_1(\alpha^{p^{-n}}))^{p^n} = 1$$

by (2.2). Thus the field of constants of $H^\infty(K)$ is contained in P. Applying (2.2) shows P is contained in the field of constants of $H^\infty(K)$, and the lemma is established.

III. Higher derivations and separating transcendency bases.

As before, K is a field of characteristic $p \neq 0$ with maximal perfect subfield P. Throughout this section we assume the transcendence degree of K/P (tr d(K/P)) is finite.

(3.1) Theorem. *Let $K \supseteq h \supseteq P$ be fields and assume K has a separating transcendency basis over h and h is algebraically closed in K. Then h is the field of constants of a set of higher derivations on K.*

Proof. Let $\{x_1, \cdots, x_n\}$ be a separating transcendency basis, and hence a relative p-basis [5, Theorem 15, p. 384], for K/h, and let T be a p-basis for h. Since K/h is separable, $\{x_1, \cdots, x_n\} \cup T$ is a p-basis for K. Define $\mathscr{F} = \{d^1, \cdots, d^n\}$ by

$$d_1^i(x_i) = 1, \qquad d_1^i(x) = 0, \qquad x \in \{x_1, \cdots, \hat{x}_i, \cdots, x_n\} \cup T, 1 \leqq i \leqq n,$$

and

$$d_j^i(x) = 0, \qquad x \in \{x_1, \cdots, x_n\} \cup T,\ 1 \leqq i \leqq n,\ 1 < j < \infty.$$

Since $d_j^i(t)=0$ $\forall t \in T$, $1 \leqq i \leqq n$, $1 \leqq j \leqq \infty$, h is contained in the field of constants of $\mathscr{F}$. By [**4**, Theorem 3.2], the transcendence degree of K over the field of constants of $\mathscr{F}$ is n, and hence h is the field of constants of $\mathscr{F}$.

(3.2) EXAMPLE. The condition of Theorem (3.1) is not necessary. Consider the following example [**5**, Example 10, p. 389]. Let P be a perfect field and $Z=\{z_1, z_2, \cdots\}$ a denumerable set of elements algebraically independent over P. Let $P(Z^{p^{-\infty}})$ be the perfect field

$$P(Z^{p^{-\infty}}) = P(Z, Z^{p^{-1}}, Z^{p^{-2}}, \cdots).$$

Let y, u_0 be algebraically independent over $P(Z^{p^{-\infty}})$, and define quantities u_n recursively by

$$u_n = y^{p^{n-1}} + Z_n u_{n-1} \qquad (n = 1, 2, \cdots).$$

Let $K=P(Z^{p^{-\infty}}, y, u_0, u_1^{p^{-1}}, u_2^{p^{-2}}, \cdots, u_n^{p^{-n}}, \cdots)$.

Mac Lane has shown $P(Z^{p^{-\infty}})$ is the maximal perfect subfield of K, K has $\{y, u_0\}$ as a transcendency basis over $P(Z^{p^{-\infty}})$, and $\{y\}$ is a p-basis for K. Thus K does not have a separating transcendency basis over $P(Z^{p^{-\infty}})$, but by (2.4), $P(Z^{p^{-\infty}})$ is the field of constants of $H^\infty(K)$.

This example also shows that not every regular subfield h of K containing P is the field of constants of a set of higher derivations. Let h be the algebraic closure of $P(Z^{p^{-\infty}}, y)$ in K. Since $\{y\}$ is p-independent in K, K/h is separable and hence regular. Since $\operatorname{tr\,d}(K/P(Z^{p^{-\infty}}))=2$, $h \neq K$. Since $\{y\}$ is a p-basis for K, the null set $\varnothing$ is a relative p-basis for K/h and hence by (2.1) $H_h^\infty(K)=\{0\}$ and h is not the field of constants of any set of higher derivations on K.

Let h be a subfield of k containing P such that K is separable over h and assume $\operatorname{tr\,d}(K/h)<\infty$.

(3.3) THEOREM. *Every regular subfield k of K containing h is the field of constants of a set of higher derivations on K if and only if K has a separating transcendency basis over h.*

PROOF. If K has a separating transcendency basis over h, then K has one over any regular subfield k containing h [**5**, Theorem 18, p. 387], and hence every regular subfield k containing h is the field of constants of a set of higher derivations (3.1). Conversely, assume K does not have a separating transcendency basis over h. Let T be any relative p-basis for K over h. Since T is algebraically independent over h, in view of [**5**, Theorem 13, p. 383], T cannot be a transcendency basis for K over h.

Thus if we let k be the algebraic closure of $h(T)$ in K, $k \neq K$, k is a regular subfield of K (since K/k preserves p-independence) and as in (3.2) $H_k^\infty(K)=\{0\}$. Thus the theorem follows.

(3.4) COROLLARY. *The following are equivalent.*

(1) *There exists a transcendency basis T for K over h such that K^{p^n} is a separable extension of $h(T)$.*

(2) *Every regular subfield of K containing h is the field of constants of a set of higher derivations on K.*

(3) *K has a separating transcendency basis over h.*

PROOF. The equivalence of (1) and (3) is [**5**, Theorem 6, p. 375]. (3.3) shows (2) equivalent to (3).

IV. Transcendence degree of $K/h=\infty$.

(4.1) THEOREM. *Let K be a field of characteristic $p \neq 0$. Let h be a subfield of K containing P such that K is separable over h and assume the transcendence degree of K over h is infinite. Then there exists a regular subfield k of K containing h which is not the field of constants of any set of higher derivations on K.*

PROOF. Let T be any relative p-basis for K over h. If $|T|<\infty$, let k be the algebraic closure of $P(T)$ in K. Then K is regular over k (K/k preserves p-independence) and since $\varnothing$ is a relative p-basis for K over k, $H_k^\infty(K)=\{0\}$ and k is the desired subfield. If $|T|=\infty$, let $T=\{x_1, x_2, \cdots\} \cup S$. Let k_1 be the algebraic closure of $h(s)$ in K. Then $\{x_1, x_2, \cdots\}$ is a relative p-base for K over k_1. Elementary calculations show $\{x_1x_2^p, x_2x_3^p, \cdots, x_nx_{n+1}^p, \cdots\}$ is also a relative p-basis. Since K/k_1 is separable, $\{x_1x_2^p, x_2x_3^p, \cdots\}$ is algebraically independent over k_1. We claim $\{x_1, x_1x_2^p, x_2x_3^p, \cdots\}$ is also algebraically independent over k_1. If not, $\{x, x_1x_2^p, \cdots, x_{n-1}x_n^p\}$ must be algebraically dependent over k_1 for some n. But $k_1(x_1, x_2, \cdots, x_n)$ is algebraic over $k_1(x_1, x_1x_2^p, \cdots, x_{n-1}x_n^p)$ and hence

$$\operatorname{tr\,d}(k_1(x_1, \ldots, x_n)/k_1) < n,$$

a contradiction. Thus $\{x_1, x_1x_2^p, \cdots\}$ is algebraically independent over k_1. Let k be the algebraic closure of $k_1(x_1x_2^p, x_2x_3^p, \cdots)$ in K. Then k is a regular subfield of K, $k \neq K$, and $\varnothing$ is a relative p-basis for K over k. Thus $H_k^\infty(K)=\{0\}$ and k is not the field of constants of any set of higher derivations on K.

(4.2) COROLLARY. *Let K be a field of characteristic $p \neq 0$. Let h be a subfield of K containing P such that K is separable over h. Then every regular subfield of K containing h is the field of constants of a set of higher* *derivations on K if and only if* (1) *the transcendence degree of K over h is finite and* (2) *K has a separating transcendency basis over h.*

The Galois theory established in [**4**] required that K be finitely generated over the distinguished regular subfields. In view of (4.2) we see that the correspondence can be extended to regular subfields h such that K has a finite separating transcendency basis over h. In part, the Galois correspondence can now be stated as follows.

(4.3) THEOREM. *Assume K has a finite separating transcendency basis over a regular subfield h containing P. Then there exists a one-to-one correspondence between the regular subfields of K containing h and Galois subgroups of* $H_h^\infty(K)$.

The characterization of the Galois subgroups remains the same as in [**4**]. Moreover, (4.2) shows the condition that K have a finite separating transcendency basis over h to be the most general we can impose and maintain a complete correspondence in that all regular subfields of K containing h will be fields of constants of sets of higher derivations.

REFERENCES

1. N. Heerema, *Convergent higher derivations on local rings*, Trans. Amer. Math. Soc. **132** (1968), 31–44. MR **36** #6406.

2. ———, *Derivations and embeddings of a field in its power series ring*. II, Michigan Math. J. **8** (1961), 129–134. MR **25** #69.

3. M. Weisfeld, *Purely inseparable extensions and higher derivations*, Trans. Amer. Math. Soc. **116** (1965), 435–449. MR **33** #122.

4. N. Heerema and J. Deveney, *Galois theory for fields K/k finitely generated*, Trans. Amer. Math. Soc. (to appear).

5. S. Mac Lane, *Modular fields*. I. *Separating transcendency bases*, Duke Math. J. **5** (1939), 372–393.

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