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ALLOWABLE DIAGRAMS FOR PURELY INSEPARABLE FIELD EXTENSIONS

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Abstract. We define a diagram associated with a purely inseparable field extension of finite exponent. We show that, under this definition, for any given field extension the shape of its diagram is unique. Thus our diagram improves the diagram defined by Sweedler in [2, p. 402].

In §2 we define an "allowable" shape for a diagram. Given any "allowable" shape for a diagram representing a finite field extension, we construct a field extension whose diagram has that shape.

1. **The definition of a diagram.** Let K be a purely inseparable field extension of k of exponent n (i.e. n is the minimal integer such that $K^{p^n}\subset k$). Consider the following chain of intermediate fields:

$$k\subset k^{1/p}\cap K\subset k^{1/p^2}\cap K\subset\cdots\subset k^{1/p^{n-1}}\cap K\subset k^{1/p^n}\cap K=K$$

where $k^{1/p^i}\cap K=\{x\in K|x^{p^i}\in k\}$. Each field has exponent one over the previous field. A diagram of K over k will be an $n\times n$ grid with sets of elements of K as entries. We will denote the set in the (i,j)th position by $S_{i,j}$. $S_{i,j}=\varnothing$ if $i<j$. In $S_{1,1}$ we place any p-basis for K over $k^{1/p^{n-1}}\cap K$. We let $S_{2,1}$ be any maximal set in $S_{1,1}^p=\{x^p|x\in S_{1,1}\}$ p-independent over $k^{1/p^{n-2}}\cap K$, and $S_{2,2}$ a completion of $S_{2,1}$ to a p-basis for $k^{1/p^{n-1}}\cap K$ over $k^{1/p^{n-2}}\cap K$. Choose any maximal set in $S_{2,1}^p$ p-independent over $k^{1/p^{n-3}}\cap K$ as $S_{3,1}$, any maximal set in $S_{2,2}^p$ p-independent over $(k^{1/p^{n-3}}\cap K)[S_{3,1}]$ as $S_{3,2}$, and any completion of $S_{3,1}\cup S_{3,2}$ to a p-basis for $k^{1/p^{n-2}}\cap K$ over $k^{1/p^{n-3}}\cap K$ as $S_{3,3}$. Continue in this manner. The process will terminate with the choice of $S_{n,n}$.

$S_{1,1}\cup S_{2,2}\cup\cdots\cup S_{n,n}$ will generate K over k. Though each $S_{i,j}$ is clearly not unique, we will show that $|S_{i,j}|$, the cardinality of the set $S_{i,j}$, is uniquely determined. We note that this is not the case in the definition of a diagram of a purely inseparable field extension as outlined by Sweedler in [2, p. 402]. The difference between the diagrams is that ours concentrates elements to the left to the extent possible.

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Set $K_{i0}=k^{1/p^{n-i}}\cap K$ for $i\leqq n=\exp(K/k)$, and $K_{ij}=K_{i,j-1}[S_{ij}]$ for $0<j\leqq i$. Then since S_{ij} is chosen to be a p-independent set over $K_{i,j-1}$, $[K_{ij}:K_{i,j-1}]=p^{|S_{ij}|}$. Hence we need only show that each K_{ij} is uniquely determined by K/k.

Now, the definition of S_{ij} guarantees that, for $j>0$:

$$\begin{aligned} K_{ij} = K_{i,j-1}[S_{ij}] &= K_{i,j-1}[S^p_{i-1,j}], && j<i, \\ &= K_{i-1,0}, && j=i. \end{aligned}$$

So we can easily prove

LEMMA 1. *$K^p_{i-1,j}\subseteq K_{ij}$ for $j<i$ and $i>0$.*

PROOF. This is clearly true for $j=0$. Proceed by induction on j, and assume true for $j-1$. Then $K_{i-1,j}=K_{i-1,j-1}[S_{i-1,j}]$, and so

$$K^p_{i-1,j} = K^p_{i-1,j-1}[S^p_{i-1,j}] \subseteq K_{i,j-1}[S^p_{i-1,j}] = K_{ij}. \quad \text{Q.E.D.}$$

Observe now that $K_{ij}=K_{i,j-1}K^p_{i-1,j}$ for all $j<i$ and $i>0$. For $K_{i,j-1}\subseteq K_{ij}$, $K^p_{i-1,j}\subseteq K_{ij}$ (by the lemma) $\Rightarrow K_{i,j-1}K^p_{i-1,j}\subseteq K_{ij}$. Conversely,

$$K_{ij} = K_{i,j-1}[S^p_{i-1,j}] \subseteq (K_{i,j-1})(K_{i-1,j-1}[S_{i-1,j}])^p = K_{i,j-1}K^p_{i-1,j}.$$

Hence the desired equality holds.

That K_{ij} is uniquely determined by K/k then follows from this equality via an easy induction agrument. Hence we have proved

THEOREM 2. *The shape of a diagram of a purely inseparable field extension of finite exponent is uniquely determined.*

2. **Allowable diagram shapes.** This section will be devoted to showing that given any "allowable" shape for a diagram representing a finite field extension we can construct a field extension whose diagram has that shape.

DEFINITION. An allowable shape for a diagram of exponent n and of finite degree will be an $n\times n$ grid with finite sets of stars as entries. We will denote the (i,j)th entry as $a_{i,j}$. The stars must obey the following distribution rules:

(1) If $i<j$, $|a_{i,j}|$ (=the cardinality of the set $a_{i,j}$)$=0$.

(2) $|a_{i,1}|\neq 0$ for $i=1,\cdots,n$.

(3) For any fixed i_0, j_0 with $i_0\geqq j_0$, $|a_{i,j_0}|\leqq|a_{i_0,j_0}|$ for all $i\geqq i_0$.

Given any allowable diagram shape D of exponent n, with entries denoted by $a_{i,j}$, we will denote by $r(D)$ the sum $|a_{1,1}|+|a_{2,2}|+\cdots+|a_{n,n}|$. We will prove our assertion that any allowable shape has a corresponding field extension by inducting on r.

Let $r=1$, and let D be any allowable diagram shape such that $r(D)=1$.

Then if D is of exponent n, D must have a diagram of the form:

$$\begin{array}{|c|}\hline * \\ \hline * \\ \hline \cdot \\ \cdot \\ \cdot \\ \hline * \\ \hline \end{array}$$

where there are n stars.

If we let Z_p designate the prime field of characteristic p, and let x be an indeterminate, then $Z_p(x)$ over $Z_p(x^{p^n})$ has diagram shape D.

Assume that for any allowable diagram shape D such that $r(D)=m>1$, we can find a corresponding field extension.

Let D_0 be any allowable diagram shape, say of exponent n, such that $r(D_0)=m+1$. Assume a_{i_0,i_0} is the last nonempty diagonal entry (i.e. $|a_{i,i}|=0$ if $i>i_0$). Let j be the maximal number $\leqq n$ such that $|a_{j,i_0}|=|a_{i_0,i_0}|$. Let D_0^- be the diagram shape D_0 stripped of one star in each of a_{i,i_0}, $i=i_0,\cdots,j$. D_0^- is nonempty, since $r(D_0)=m+1$, and $m>1$. A quick check of the rules will verify that D_0^- is an allowable shape of exponent n. $r(D_0^-)=m\,(>1)$ so by the induction hypothesis there exists a field extension say K/k with diagram shape D_0^-. From this K we wish to construct a field extension $\mathscr{L}$ over l which has diagram shape D_0. To do this we need the following two lemmas.

Lemma 3. *If K/k has diagram shape D, x an indeterminate, then $K(x)/k(x)$ has diagram shape D.*

Proof. First we prove that $k(x)^{1/p^n}\cap K(x)=(k^{1/p^n}\cap K)(x)$. Clearly $(k^{1/p^n}\cap K)(x)\subset k(x)^{1/p^n}\cap K(x)$. We will show we have the opposite inclusion. Consider any $w=f(x)/g(x)\in k(x)^{1/p^n}\cap K(x)$, where $f(x)$, $g(x)\in K[x]$. We may assume that $f(x)/g(x)$ is in lowest terms, that is $f(x)$, $g(x)$ are monic and relatively prime in $K[x]$. $f(x)/g(x)$ is a unique representation of w. We have that

$$w^{p^n}=[f(x)/g(x)]^{p^n}=[f(x)]^{p^n}/[g(x)]^{p^n}\in k(x).$$

Since $[f(x)]^{p^n}/[g(x)]^{p^n}$ is again in lowest terms, this is a unique representation of w^{p^n} in $K(x)$. Hence $[f(x)]^{p^n}$, $[g(x)]^{p^n}\in k[x]\Rightarrow$ the coefficients of $f(x)$, $g(x)$ are in $k^{1/p^n}\cap K$. Hence, $w\in(k^{1/p^n}\cap K)(x)$.

Any p-basis for $k^{1/p^{n+1}}\cap K$ over $k^{1/p^n}\cap K$ is a p-basis for $(k^{1/p^{n+1}}\cap K)(x)$ over $(k^{1/p^n}\cap K)(x)$, and hence for $k(x)^{1/p^{n+1}}\cap K(x)$ over $k(x)^{1/p^n}\cap K(x)$. So any diagram for K over k remains a diagram for $K(x)/k(x)$. Q.E.D.

LEMMA 4. *Let K be a purely inseparable extension of k, let x, y be indeterminates and let z be an element in K of exponent s over k. Set $l=k(x^{p^t}, y^{p^t})$, $L=K(x^{p^t}, y^{p^t})$ where $t\leqq s$. Then*

$$\begin{aligned} l^{1/p^n} \cap L[xz+y] &= (l^{1/p^n} \cap L)[xz+y] && \text{if } n \geqq s, \\ &= (l^{1/p^n} \cap L)[(xz+y)^{p^{s-n}}] && \text{if } s-t<n<s, \\ &= l^{1/p^n} \cap L && \text{if } n \leqq s-t. \end{aligned}$$

PROOF. We have that

$$\begin{aligned} l^{1/p^n} \cap L[xz+y] &\subset k(x,y)^{1/p^n} \cap K(x,y) \\ &= (k^{1/p^n} \cap K)(x,y) \quad \text{(by Lemma 3)} \\ &\subset (l^{1/p^n} \cap L)[x,y]. \end{aligned}$$

Consider $w \in l^{1/p^n} \cap L[xz+y]$. Since $xz+y$ has exponent t over L, we can write w as $\sum_{i=0}^{p^t-1} \alpha_i(xz+y)^i$, for some $\alpha_i \in L$. Hence

$$\begin{aligned} w = \alpha_0 + (\alpha_1 zx + \alpha_1 y) + \sum_{j=0}^{2} \left(\alpha_2 \binom{2}{j} z^j\right) x^j y^{2-j} \\ + \cdots + \sum_{j=0}^{p^t-1} \left(\alpha_{p^t-1} \binom{p^t-1}{j} z^j\right) x^j y^{(p^t-1)-j}, \qquad w \in (l^{1/p^n} \cap L)[x,y], \end{aligned}$$

$\Rightarrow$ coefficients of the above equation considered as a polynomial in x and y (i.e. the various $\alpha_i\binom{i}{j}z^j$) lie in $l^{1/p^n} \cap L$. In particular then $\{\alpha_i\binom{i}{0}z^0\}$ $(i=0,\cdots,p^t-1) = \{\alpha_0, \alpha_1, \cdots, \alpha_{p^t-1}\}$ lies in $l^{1/p^n} \cap L$. So

$$w \in (l^{1/p^n} \cap L)[xz+y].$$

Therefore $l^{1/p^n} \cap L[xz+y] \subset (l^{1/p^n} \cap L)[xz+y]$. We will use this inclusion in each of the following cases.

Case $n \geqq s$. Since $(xz+y)^{p^n} \in l$ if $n \geqq s$, then $(xz+y) \in l^{1/p^n} \cap L[xz+y]$. Clearly $(l^{1/p^n} \cap L) \subset l^{1/p^n} \cap L[xz+y]$. So

$$\begin{aligned} (l^{1/p^n} \cap L)[xz+y] &\subset l^{1/p^n} \cap L[xz+y] \\ &\Rightarrow (l^{1/p^n} \cap L)[xz+y] = l^{1/p^n} \cap L[xz+y]. \end{aligned}$$

Case $n<s$. Set $u=xz+y$; we shall show that

$$l^{1/p^n} \cap L(u) = (l^{1/p^n} \cap L)(u^{p^{s-n}})$$

for $n<s$ (note that, if $n \leqq s-t$, then $s-n \geqq t \Rightarrow u^{p^{s-n}}$ is in $l^{1/p^n} \cap L$, and so the RHS of the equality is $l^{1/p^n} \cap L$).

We claim first that the exponent of u over $(l^{1/p^n} \cap L)(u^{p^{s-n}})$ is $s-n$. For $u^{p^{s-n-1}}$ in $(l^{1/p^n} \cap L)(u^{p^{s-n}}) \Rightarrow u^{p^{s-1}}$ is in $l(u^{p^s})=l$, contradicting the fact that u has exponent s over l. Thus

$$[(l^{1/p^n} \cap L)(u):(l^{1/p^n} \cap L)(u^{p^{s-n}})] = p^{s-n}.$$

Now, given w in $l^{1/p^n} \cap L(u) \subseteq (l^{1/p^n} \cap L)(u)$, we may write $w = \sum_{i=0}^{p^{s-n}-1} a_i u^i$, with a_i in $(l^{1/p^n} \cap L)(u^{p^{s-n}})$. Then $w^{p^n} = \sum_{i=0}^{p^{s-n}-1} a_0^{p^n} u^{ip^n}$ is in l, and each $a_i^{p^n}$ is in l. Moreover, the set $\{u^{ip^n} | i=0, \cdots, p^{s-n}-1\}$ is linearly independent over l, since u has exponent s over l. We may then conclude that $a_i = 0$ for all $i > 0$, and so $w = a_0$ is in $(l^{1/p^n} \cap L)(u^{p^{s-n}})$, completing the proof. Q.E.D.

Now we must show that $\mathscr{L} = L[xz+y]$ has diagram shape D_0 over l.

(a) If $n \geqq s$ we have that any p-basis for $l^{1/p^{n+1}} \cap L$ over $l^{1/p^n} \cap L$ is p-independent over $(l^{1/p^n} \cap L)[xz+y] = l^{1/p^n} \cap L[xz+y]$ because $xz+y$ has exponent t over both $l^{1/p^n} \cap L$ and $l^{1/p^{n+1}} \cap L$. It generates

$$(l^{1/p^{n+1}} \cap L)[xz + y] = l^{1/p^{n+1}} \cap L[xz + y]$$

over $l^{1/p^n} \cap L[xz+y]$, and so is a p-basis for

$$l^{1/p^{n+1}} \cap L[xz + y]/l^{1/p^n} \cap L[xz + y].$$

(b) If $s-t \leqq n < s$ and $\{w\}$ is any p-basis for $l^{1/p^{n+1}} \cap L$ over $l^{1/p^n} \cap L$, then $\{w\} \cup \{(xz+y)^{p^{s-(n+1)}}\}$ is a p-basis for $l^{1/p^{n+1}} \cap L[xz+y] = (l^{1/p^{n+1}} \cap L) \cdot [(xz+y)^{p^{s-(n+1)}}]$ over $l^{1/p^n} \cap L[xz+y] = (l^{1/p^n} \cap L)[(xz+y)^{p^{s-n}}]$.

(c) If $n \leqq s-t$, we have that $l^{1/p^n} \cap L[xz+y] = l^{1/p^n} \cap L$, so trivially any p-basis for $l^{1/p^n} \cap L$ over $l^{1/p^{n-1}} \cap L$ is a p-basis for $l^{1/p^n} \cap L[xz+y]$ over $l^{1/p^{n-1}} \cap L[xz+y]$.

Thus if we choose a z from K such that the exponent of z over k is $n-i_0+1$, and we set $t=j-i_0+1$, $K(xz+y, x^{p^t}, y^{p^t})$ will have diagram shape D_0 over $k(x^{p^t}, y^{p^t})$. So we may finally conclude

THEOREM 5. *Given any allowable shape for a diagram representing a finite field extension, there exists a purely inseparable field extension whose diagram has that shape.*

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