IDENTITIES IN IMPLICATIVE SEMILATTICES

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Abstract. An effective procedure is given for deciding whether or not an equation in the theory of implicative semilattices is an identity.

An implicative semilattice (Curry [1]) is a nonempty set \( S \) together with a binary relation \( \leq \) in \( S \), and two binary operations \( \land \) and \( \rightarrow \) in \( S \), satisfying, for all \( a, b, c \) in \( S \):

1. \( a \leq a \).
2. If \( a \leq b \) and \( b \leq c \), then \( a \leq c \).
3. \( a \land b \leq a \) and \( a \land b \leq b \).
4. If \( c \leq a \) and \( c \leq b \), then \( c \leq a \land b \).
5. If \( a \land b \leq c \), then \( a \leq c \land b \).
6. \( a \land (a \rightarrow b) \leq b \).
7. If \( a \leq b \) and \( b \leq a \), then \( a = b \).

\( \leq \) is a partial order by (1), (2) and (7); \( a \land b \) is the greatest lower bound of \( \{a, b\} \) by (3) and (4).

Let \( a, b, c, d, e \) be elements of an implicative semilattice. The following properties are easily derived.

8. \( (a \rightarrow d) \land (a \rightarrow e) = a \rightarrow (d \land e) \).
9. \( (a \land d) \rightarrow e = a \rightarrow (d \rightarrow e) \).
10. If \( a \rightarrow c \leq a \), then \( a \rightarrow c \leq c \).

A term is a meaningful expression built up from variables, \( \land \) and \( \rightarrow \) and parentheses. The rank of a term is the number of occurrences of \( \land \) and \( \rightarrow \) in the term. If \( a \) and \( b \) are terms, then \( a = b \) is an equation. An equation is an identity if and only if it holds in every implicative semilattice, or equivalently, if it follows from (1) through (7). The purpose of this paper is to give an effective procedure for deciding, for any terms \( a \) and \( b \), whether or not the equation \( a = b \) is an identity.

It is easy to see that the problem of whether or not \( a = b \) is an identity reduces to the problem of whether or not \( a \leq b \) and \( b \leq a \) are derivable from (1) through (6).

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The decision procedure makes use of a relation $\subset$, which is not set inclusion, but an extension of $\leq$. We first set the notation. $a, b, c, d, e$ are terms. $v$ is a variable. $R, S, T, U$ are finite (perhaps empty) sets of terms. $(S, T)$ is the union of $S$ and $T$. $(S, a, b)$ is the union of $S$ and $\{a, b\}$.

**Definition.** $S \subset c$ if and only if it is derivable from $R_1$ through $R_7$:

- **R1.** $(S, v) \subset v$.
- **R2.** If $(S, b) \subset c$ and $(S, b) \subset d$, then $(S, b) \subset c \land d$.
- **R3.** If $(S, b, a) \subset c$, then $(S, b) \subset a \rightarrow c$.
- **R4.** If $(R, d, e) \subset P$, then $(R, d \land e) \subset v$.
- **R5.** If $(R, a \rightarrow d, a \rightarrow e) \subset v$, then $(R, a \rightarrow (d \land e)) \subset v$.
- **R6.** If $(R, (a \land d) \rightarrow e) \subset v$, then $(R, a \rightarrow (d \rightarrow e)) \subset v$.
- **R7.** If $(R, a \rightarrow v) \subset a$, then $(R, a \rightarrow v) \subset v$.

**Lemma A.** If $T \subset c$, then $(T, U) \subset c$.

**Proof.** In a given derivation of $T \subset c$, replace each instance $(S, v) \subset v$ of R1 by $(S, U, v) \subset v$. R2 through R7 carry $U$ along, so that the given derivation of $T \subset c$ becomes *mutatis mutandis* a derivation of $(T, U) \subset c$.

**Lemma B.** If $T \subset b$ and $(S, b) \subset c$, then $(T, S) \subset c$.

**Proof.** The proof is by a double strong induction. The primary induction is on the rank of $b$. The secondary induction is on the number of applications of R1 through R7 in a given derivation of $(S, b) \subset c$. We first consider two special cases whose proofs do not depend on the induction framework.

**Case 0.** $b$ is in $S$. Then $(S, b) \subset c$ is $S \subset c$, and Lemma A gives $(T, S) \subset c$. From now on it is assumed that Case 0 does not hold.

**Case 1.** $(S, b) \subset c$ by R1. Then $c$ is a variable, and either $c$ is $b$ or $S$ is $(S_1, c)$. If $c$ is $b$, then Lemma A applied to $T \subset b$ gives $(T, S) \subset b$. If $S$ is $(S_1, c)$, then R1 gives $(T, S_1, c) \subset c$.

**Basis.** Suppose the rank of $b$ is zero, and the number of applications of R1 through R7 in the given derivation of $(S, b) \subset c$ is one. From the latter it follows that $(S, b) \subset c$ is inferred by R1, and Case 1 applies.

**Induction step.** Seven cases are considered, according as $(S, b) \subset c$ is inferred by Ri, $1 \leq i \leq 7$. Case 1 takes care of R1.

**Case 2.** $(S, b) \subset c$ is inferred by R2. Then $c$ is $c_1 \land c_2$, and $(S, b) \subset c_1 \land c_2$ is inferred from $(S, b) \subset c_1$ and $(S, b) \subset c_2$. The SIH (secondary induction hypothesis) applied to $T \subset b$ and $(S, b) \subset c_1$ gives $(T, S) \subset c_1$. Similarly $(T, S) \subset c_2$. Then R2 gives $(T, S) \subset c_1 \land c_2$.

**Case 3.** $(S, b) \subset c$ is inferred by R3. Then $c$ is $c_1 \rightarrow c_2$, and $(S, b) \subset c_1 \rightarrow c_2$ is inferred from $(S, b, c_1) \subset c_2$. The SIH applied to $T \subset b$ and $(S, b, c_1) \subset c_2$ gives $(T, S, c_1) \subset c_2$, and then R3 gives $(T, S) \subset c_1 \rightarrow c_2$. 

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The conclusion of $R_i$, $4 \leq i \leq 7$, is $(R, t) = v$, where $t$ is a term. Suppose $(S, b) \leq c$ is inferred by $R_i$, $4 \leq i \leq 7$. Then $c$ is a variable, and two cases arise, according as $b$ is or is not $t$. If $b$ is $t$ (and $S$ is $R$), then Case $i$ below applies. If $b$ is not $t$, then the proof follows the pattern of Cases 2 and 3: first apply the SIH, and then apply $R_i$. We illustrate with $R_4$. Suppose $(S, b) \leq c$ is inferred by $R_4$, and $b$ is not the term $dAe$ that explicitly occurs in $R_4$. Then $S$ is $(S_1, d \land e)$, and $(S_1, d \land e, b) \leq c$ is inferred from $(S_1, d, e, b) \leq c$. The SIH applied to $T \leq b$ and $(S_1, d, e, b) \leq c$ gives $(T, S_1, d, e) \leq c$, and then $R_4$ gives $(T, S_1, d \land e) \leq c$.

Case 4. $(S, b) \leq c$ is inferred by $R_4$ and $b$ is $d \land e$ and $S$ is $R$. Then $(S, d \land e) \leq c$ is inferred from $(S, d, e) \leq c$. $T \leq b$ is $T \leq d \land e$, and is necessarily inferred by $R_2$ from $T \leq d$ and $T \leq e$. The PIH (primary induction hypothesis) applied to $T \leq d$ and $(S, d, e) \leq c$ gives $(T, S, e) \leq c$. Then the PIH applied to $T \leq e$ and $(T, S, e) \leq c$ gives $(T, S) \leq c$.

Case 5. $(S, b) \leq c$ is inferred by $R_5$, and $b$ is $a \rightarrow (d \land e)$ and $S$ is $R$. Then $(S, a \rightarrow (d \land e)) \leq c$ is inferred from $(S, a \rightarrow d, a \rightarrow e) \leq c$. $T \leq b$ is $T \leq a \rightarrow (d \land e)$, and is necessarily inferred from $(T, a) \leq d \land e$ by $R_3$. The latter is necessarily inferred by $R_2$ from $(T, a) \leq d$ and $(T, a) \leq e$. $R_3$ applied to $(T, a) \leq d$ gives $T \leq a \rightarrow d$. Similarly $T \leq a \rightarrow e$. The PIH applied to $T \leq a \rightarrow d$ and $(S, a \rightarrow d, a \rightarrow e) \leq c$ gives $(T, S, a \rightarrow e) \leq c$. Then the PIH applied to $T \leq a \rightarrow e$ and $(T, S, a \rightarrow e) \leq c$ gives $(T, S) \leq c$.

Case 6. $(S, b) \leq c$ is inferred by $R_6$, and $b$ is $a \rightarrow (d \rightarrow e)$ and $S$ is $R$. Then $(S, a \rightarrow (d \rightarrow e)) \leq c$ is inferred from $(S, (a \land d) \rightarrow e) \leq c$. $T \leq b$ is $T \leq a \rightarrow (d \rightarrow e)$, and is necessarily inferred by $R_3$ from $(T, a) \leq d \rightarrow e$. The latter is necessarily inferred by $R_3$ from $(T, a, d) \leq e$. $R_4$ gives $(T, a \land d) \leq e$, and then $R_3$ gives $T \leq (a \land d) \rightarrow e$. The SIH applied to $T \leq (a \land d) \rightarrow e$ and $(S, (a \land d) \rightarrow e) \leq c$ gives $(T, S) \leq c$.

Case 7. $(S, b) \leq c$ is inferred by $R_7$, and $b$ is $a \rightarrow c$ and $S$ is $R$. Then $(S, a \rightarrow c) \leq c$ is inferred from $(S, a \rightarrow c) \leq a$. $T \leq b$ is $T \leq a \rightarrow c$. The SIH applied to $T \leq a \rightarrow c$ and $(S, a \rightarrow c) \leq a$ gives $(T, S) \leq a$. $T \leq a \rightarrow c$ is necessarily inferred by $R_3$ from $(T, a) \leq c$. The PIH applied to $(T, S) \leq a$ and $(T, a) \leq c$ gives $(T, S) \leq c$.

**Lemma C.** If $(S, d, e) \leq c$, then $(S, d \land e) \leq c$.

**Lemma D.** If $(S, a \rightarrow d, a \rightarrow e) \leq c$, then $(S, a \rightarrow (d \land e)) \leq c$.

**Lemma E.** If $(S, (a \land d) \rightarrow e) \leq c$, then $(S, a \rightarrow (d \rightarrow e)) \leq c$.

**Proofs.** Lemma D is proved by strong induction on the rank $n$ of $c$. If $n = 0$, then $c$ is a variable $v$, and $R_5$ gives the desired result. For the induction step, we consider two cases, according as $c$ is $c_1 \land c_2$ or $c$ is $c_1 \rightarrow c_2$. Suppose $c$ is $c_1 \land c_2$, and $(S, a \rightarrow d, a \rightarrow e) \leq c_1 \land c_2$. The latter is
necessarily inferred by \(R2\) from \((S, a\rightarrow d, a\rightarrow e)\subseteq c_1\) and \((S, a\rightarrow d, a\rightarrow e)\subseteq c_2\). The induction hypothesis gives \((S, a\rightarrow (d\land e))\subseteq c_1\) and \((S, a\rightarrow (d\land e))\subseteq c_2\), and then \(R2\) gives \((S, a\rightarrow (d\land e))\subseteq c_1\land c_2\). Suppose \(c\) is \(c_1\subseteq c_2\), and \((S, a\rightarrow d, a\rightarrow e)\subseteq c_1\rightarrow c_2\). The latter is necessarily inferred by \(R3\) from \((S, a\rightarrow d, a\rightarrow e, c_1)\subseteq c_2\). The induction hypothesis gives \((S, a\rightarrow (d\land e), c_1)\subseteq c_2\), and then \(R3\) gives \((S, a\rightarrow (d\land e))\subseteq c_1\rightarrow c_2\). Lemmas C and E are proved in exactly the same way, with the basis steps given by \(R4\) and \(R6\) respectively.

**Lemma F.** \((S, c)\subseteq c\). If \(T\subseteq a\), then \((T, a\rightarrow c)\subseteq c\).

**Proof.** The two parts are proved simultaneously by strong induction on the rank \(n\) of \(c\). If \(n=0\), then \(c\) is a variable \(v\). \(R1\) gives \((S, v)\subseteq v\).

Suppose \(T\subseteq a\). Then Lemma A gives \((T, a\rightarrow v)\subseteq a\), and \(R7\) gives \((T, a\rightarrow v)\subseteq v\). For the induction step we consider two cases, according as \(c\) is \(c_1\land c_2\) or \(c\) is \(c_1\rightarrow c_2\). Suppose \(c\) is \(c_1\land c_2\). The induction hypothesis gives \((S, c_1, c_2)\subseteq c_1\land c_2\) and \((S, c_1, c_2)\subseteq c_2\), and then \(R2\) gives \((S, c_1, c_2)\subseteq c_1\land c_2\), and then Lemma C gives \((S, c_1\land c_2)\subseteq c_1\land c_2\). Suppose \(T\subseteq a\). The induction hypothesis gives \((T, a\rightarrow c_1)\subseteq c_1\). Lemma A gives \((T, a\rightarrow c_1, a\rightarrow c_2)\subseteq c_1\). Similarly \((T, a\rightarrow c_1, a\rightarrow c_2)\subseteq c_1\land c_2\), and then Lemma D gives \((T, a\rightarrow (c_1\land c_2))\subseteq c_1\land c_2\). Suppose \(c\) is \(c_1\rightarrow c_2\). The induction hypothesis gives \((S, c_1)\subseteq c_1\). Then the induction hypothesis gives \((S, c_1, c_1\rightarrow c_2)\subseteq c_2\), and then \(R3\) gives \((S, c_1\rightarrow c_2)\subseteq c_1\rightarrow c_2\). Suppose \(T\subseteq a\). Lemma A gives \((T, c_1)\subseteq a\). The induction hypothesis gives \((T, c_1)\subseteq c_1\). Then \(R2\) gives \((T, c_1)\subseteq a\land c_1\). Then the induction hypothesis gives \((T, c_1, (a\land c_1)\rightarrow c_2)\subseteq c_2\). \(R3\) gives \((T, (a\land c_1)\rightarrow c_2)\subseteq c_1\rightarrow c_2\), and then Lemma E gives \((T, a\rightarrow (c_1\rightarrow c_2))\subseteq c_1\rightarrow c_2\).

**Lemma G.** If \(a\subseteq b\), then \(a\subseteq b\). (\(a\subseteq b\) means \(\{a\}\subseteq b\).)

**Proof.** It suffices to show that (1) through (6) continue to hold if \(\subseteq\) is replaced throughout by \(\subseteq\). For (1), Lemma F gives \(a\subseteq a\). Lemma B, with \(T=\{a\}\) and \(S\) empty, gives (2). For (3), Lemma F gives \(\{a, b\}\subseteq a\), and then Lemma C gives \(a\land b\subseteq a\). Similarly \(a\land b\subseteq b\). \(R2\) gives (4), and \(R3\) gives (5). For (6), Lemma F gives \(a\subseteq a\). Lemma F gives \(\{a, a\rightarrow b\}\subseteq b\), and then Lemma C gives \(a\land (a\rightarrow b)\subseteq b\).

**Lemma H.** If \(\{a_1, \cdots, a_k\}\subseteq b\), then \(a_1\land\cdots\land a_k\subseteq b\). In particular, if \(a\subseteq b\), then \(a\subseteq b\).

**Proof.** It suffices to show that \(R1\) through \(R7\) continue to hold if \(\{a_1, \cdots, a_k\}\subseteq b\) is replaced throughout by \(a_1\land\cdots\land a_k\subseteq b\). (3) gives \(R1\) if \(S\) is not empty, and (1) gives \(R1\) if \(S\) is empty. (4) gives \(R2\). (5) gives \(R3\). \(R4\) is obvious. (8) gives \(R5\); (9) gives \(R6\). \(R7\) follows from (10).
Theorem. There is an effective procedure for deciding, for any terms
a and b in the theory of implicative semilattices, whether or not the equation
a = b is an identity.

Proof. By a previous remark, the problem reduces to whether or not
a ≤ b and b ≤ a. By Lemmas G and H, a ≤ b if and only if a ≤ b. Let Rk*,
2 ≤ k ≤ 7, be the inverse of Rk. (Interchange the if and then parts.) The
procedure is to start with a ≤ b and work backwards via the inverse rules
in an attempt to find a derivation of a ≤ b. Consider a derivation to be in
tree form. The tops are all instances of R1; the bottom is a ≤ b; R2 gives
rise to branching. Call S ≤ c an inequality. A branch is a finite sequence
I1, · · · , In of inequalities such that for each j, 1 ≤ j < n, Ij+1 is inferred
from Ij by one of R2 through R7. (In the case of R2, Ij+1 is inferred
from Ij and one other inequality.) A derivation is irredundant if and only
if no inequality occurs more than once in the same branch. (The
inequalities S ≤ c and T ≤ d are the same if and only if c is d and S is
the same set as T.) Clearly it suffices to consider only irredundant
derivations.

Starting with a ≤ b, a mixture of R2* and R3* gives rise to one or more
branches topped by inequalities of the form S ≤ v, where v is a variable.
Consider one such branch. A mixture of R4* through R6* reduces each
term on the left to one or more terms of the form u or a → u, where u is a
variable. If one of these terms is v, then R1 applies and the branch termi-
nates. If none of these terms is v, but one of them is of the form a → v,
then R7* applies, the term a moves to the right, and a new cycle begins.
If neither R1 nor R7* applies, then the branch aborts. It may happen
that R7* applies in more than one way, giving rise to alternate possible
derivations. Each possible derivation must be pursued until one derivation
terminates or all possible derivations abort.

It remains to show that the procedure terminates, i.e. each possible
derivation is finite, and there are only finitely many possible derivations.
Since only irredundant derivations need be considered, it suffices to show
that only finitely many inequalities can be generated by starting with a ≤ b
and applying the inverse rules. To this end, let v1, · · · , vk be all the
distinct variables that occur in a or b, and let n be the greater of the ranks
of a and b. Let A be the set of all terms of rank not exceeding n which
contain no variable not in the list v1, · · · , vk. Clearly A is a finite set,
say with m elements. Call an inequality S ≤ c an A-inequality if c and every
term in S are in A. Inspection shows that a ≤ b is an A-inequality, and each
inverse rule applied to an A-inequality yields one or two A-inequalities.
Hence only A-inequalities can occur in any possible derivation of a ≤ b.
In an A-inequality S ≤ c, there are m choices for c since c is in A, and there
are $2^m$ choices for $S$ since $S$ is a finite subset of $A$. Hence there are only $(m)(2^m)$ $A$-inequalities, and we are done.

The decision procedure above is based on ideas of Gentzen [2] as set forth in Kleene [3]. Modifications have been made to increase speed and efficiency by minimizing branching and alternate derivations. The chief novelty is that Gentzen's two-premise left $\rightarrow$ rule is replaced by the one-premise rule R7.

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