PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 41, Number 2, December 1973

ON INVARIANT LINEAR MANIFOLDS[1]

P. A. FILLMORE[2]

ABSTRACT. For a linear transformation A on a Banach space, let $\mathscr{L}(A)$ be the lattice of (not necessarily closed) invariant subspaces of A. For A bounded it is shown that if $\mathscr{L}(A\oplus A)\subset\mathscr{L}(T\oplus T)$, or if $\mathscr{L}(A)\subset\mathscr{L}(T)$ and T commutes with A, then T is a polynomial in A. In the case of a Hilbert space, if $\mathscr{L}(A)\subset\mathscr{L}(A^*)$ then A^* is a polynomial in A.

Introduction. A linear transformation T on a vector space V is *algebraic* if there is a nonzero polynomial p such that $p(T)=0$; it is *locally algebraic* if for each $x\in V$ there is a nonzero polynomial p (depending on x) such that $p(T)x=0$. A locally algebraic transformation need not be algebraic, but Kaplansky has shown [**4**, Theorem 15] that a bounded locally algebraic transformation on a Banach space must be algebraic. In this note we consider extensions of this fact and some related matters.

1. Specifically, let A and T be linear transformations on V such that T is locally a polynomial in A; that is, for each $x\in V$ there is a polynomial p (depending on x) with $Tx=p(A)x$. Must T then be a polynomial in A? This question may be reformulated as follows: For any linear transformation S on V, let

$$\mathscr{L}(S)=\{M\mid M \text{ is a subspace with } SM\subset M\},$$

the lattice of invariant subspaces of S. Now observe that T is locally a polynomial in A if and only if $\mathscr{L}(A)\subset\mathscr{L}(T)$ (for if $AM\subset M$, then $p(A)M\subset M$ for all polynomials p, so that when T is locally a polynomial in A we have $TM\subset M$; on the other hand, if $x\in V$ and

$$M_x=\{p(A)x\mid p \text{ is a polynomial}\},$$

then $x\in M_x\in\mathscr{L}(A)$, so that when $\mathscr{L}(A)\subset\mathscr{L}(T)$ we get $x\in M_x\in\mathscr{L}(T)$, $Tx\in M_x$, and $Tx=p(A)x$ for some polynomial p). Thus our question becomes: does $\mathscr{L}(A)\subset\mathscr{L}(T)$ imply that T is a polynomial in A? The

Received by the editors February 12, 1971.
AMS (MOS) subject classifications (1970). Primary 47A15, 47A65.
Key words and phrases. Invariant subspace lattice, locally algebraic operator.
[1] Research partially supported by a grant from the National Science Foundation.
[2] This work was done while the author was a visiting professor at the University of Toronto.

answer is no, even for bounded transformations on a Banach space. For a simple example, take

$$A=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \quad\text{and}\quad T=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$$

on a two-dimensional space.

Motivated by the fact that a locally algebraic transformation is algebraic on each finite-dimensional subspace, we next ask whether T must be a polynomial in A whenever it is a polynomial in A on each finite-dimensional subspace. Again this is false, but we will show that the analogue of Kaplansky's Theorem holds. Actually a little more is true:

THEOREM 1. *Let A and T be linear transformations on a Banach space V, with A bounded. If T is a polynomial in A on each two-dimensional subspace of V, then T is a polynomial in A.*

Before proving this, a few remarks are in order. For any linear transformation S on V, let $S^{(2)}=S\oplus S$ acting on $V\oplus V$. Then, as above, T is a polynomial in A on each two-dimensional subspace if and only if $\mathscr{L}(A^{(2)})\subset\mathscr{L}(T^{(2)})$. For a family $\mathscr{S}$ of linear transformations, let

$$\mathscr{S}^{(2)}=\{S^{(2)}\mid S\in\mathscr{S}\},\quad\text{and}$$
$$\mathscr{L}(\mathscr{S})=\{M\mid M\in\mathscr{L}(S)\text{ for all }S\in\mathscr{S}\}.$$

A general question of the type contemplated in the theorem goes as follows: For which algebras $\mathscr{A}$ of linear transformations is it true that $\mathscr{L}(\mathscr{A}^{(2)})\subset\mathscr{L}(T^{(2)})$ implies $T\in\mathscr{A}$? (For Theorem 1 take $\mathscr{A}$ to be all polynomials in A.)

LEMMA. *If $\mathscr{A}$ is an algebra with a separating vector, then $\mathscr{L}(\mathscr{A}^{(2)})\subset\mathscr{L}(T^{(2)})$ implies $T\in\mathscr{A}$.*

PROOF. Let x_0 be a separating vector, so that $A\in\mathscr{A}$ and $Ax_0=0$ imply $A=0$. Fix $A_0\in\mathscr{A}$ with $Tx_0=A_0x_0$. If y is any vector, by hypothesis there exists $A\in\mathscr{A}$ such that $Tx_0=Ax_0$ and $Ty=Ay$. Then $(A-A_0)x_0=0$ so $A=A_0$ and $Ty=A_0y$. Since y is arbitrary, $T=A_0$, as required.

PROOF OF THEOREM. We need only show that for any bounded linear transformation A on a Banach space, the algebra of polynomials in A has a separating vector. Suppose first that A is locally algebraic, so that A is algebraic by Kaplansky's theorem. If m is the minimum polynomial of A, it is easy to see (cf. Kaplansky's proof) that there is a vector x_0 at which the local minimum polynomial is m. Thus if $p(A)x_0=0$ then m divides p, and so $p(A)=0$. Hence x_0 is the required separating vector. If A is not locally algebraic, there is a vector x_0 such that $p(A)x_0=0$ implies $p=0$, and again we have a separating vector.

REMARKS. (i) As the proof shows, the theorem remains true without the topological hypotheses except when A is locally algebraic but not algebraic. To see that this case is an exception, let $V=V_2\oplus V_3\oplus\cdots$ (algebraic direct sum) with V_n an n-dimensional vector space, let J_n be the $n\times n$ Jordan cell

$$\begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & 1 & 0 \end{bmatrix}$$

acting in V_n, and let $A=J_2\oplus J_3\oplus\cdots$. Then

$$T = J_2 \oplus (J_3 + J_3^2) \oplus (J_4 + J_4^2 + J_4^3) \oplus \cdots$$

satisfies $\mathscr{L}(A^{(n)})\subset\mathscr{L}(T^{(n)})$ for all n, and yet T is not a polynomial in A.

(ii) Recall the question raised before the lemma: In what circumstances does $\mathscr{L}(\mathscr{A}^{(2)})\in\mathscr{L}(T^{(2)})$ imply $T\in\mathscr{A}$? For $n\geqq 1$ the *n-closure* of $\mathscr{A}$ is defined by

$$C_n(\mathscr{A}) = \{T \mid \mathscr{L}(\mathscr{A}^{(n)}) \subset \mathscr{L}(T^{(n)})\}$$

and the *strict closure* by

$$C_\infty(\mathscr{A}) = \bigcap_{n=1}^{\infty} C_n(\mathscr{A}).$$

Obviously $C_1(\mathscr{A})\supset C_2(\mathscr{A})\supset\cdots\supset C_\infty(\mathscr{A})\supset\mathscr{A}$. If $\mathscr{A}'$ is the commutant of $\mathscr{A}$ (the algebra of linear transformations that commute with every member of $\mathscr{A}$), and $\mathscr{A}''$ the commutant of $\mathscr{A}'$, one easily shows the additional relation $\mathscr{A}''\supset C_2(\mathscr{A})$. With this notation the question under consideration concerns the validity of the equation $C_2(\mathscr{A})=\mathscr{A}$. This can be conveniently split into $C_2(\mathscr{A})=C_\infty(\mathscr{A})$ and $C_\infty(\mathscr{A})=\mathscr{A}$. The first of these has received some study. For example, Jacobson has shown that if $C_2(\mathscr{A})$ is the algebra of all linear transformations, then so is $C_\infty(\mathscr{A})$ [**6**, p. 60]. More generally, if $\mathscr{A}$ is completely reducible (every invariant subspace has an invariant complement), then $\mathscr{A}''=C_\infty(\mathscr{A})$ [**1**, §4, Theorem 1], so that $C_2(\mathscr{A})=C_\infty(\mathscr{A})$ by the observation above.

2. We now consider several situations in which it follows from $\mathscr{L}(\mathscr{A})\subset\mathscr{L}(T)$ that T is a polynomial in A. For finite-dimensional spaces, a complete analysis of transformations A with this property is carried out in [**3**].

For example, let A be a bounded linear transformation on a Banach space, and suppose that A is similar to a transformation of the form

$B \oplus B$. If $\mathscr{L}(A) \subset \mathscr{L}(T)$, it is easy to see that T must be simultaneously similar to a transformation of the form $S \oplus S$, so that $\mathscr{L}(B^{(2)}) \subset \mathscr{L}(S^{(2)})$, and the theorem implies that T is a polynomial in A.

In another direction, it is known [**2**, Theorem 10] that on a finite-dimensional space, if $\mathscr{L}(A) \subset \mathscr{L}(T)$ and T commutes with A, then T is a polynomial in A.

THEOREM 2. *Let A and T be linear transformations on a Banach space, with A bounded. If $\mathscr{L}(A) \subset \mathscr{L}(T)$ and T commutes with A, then T is a polynomial in A.*

PROOF. Suppose first that A is algebraic. For any vectors x and y, let $M_{x,y}$ be the subspace consisting of all $r(A)x + s(A)y$, where r and s are polynomials. Then $M_{x,y}$ is invariant for both A and T, and their restrictions continue to satisfy the hypotheses. Since $M_{x,y}$ is finite-dimensional there is by [**2**, Theorem 10] a polynomial p such that $T = p(A)$ on $M_{x,y}$. In particular, $T = p(A)$ on the span of x and y. Because x and y are arbitrary, Theorem 1 implies that T is a polynomial in A.

If A is not algebraic, then it is not locally algebraic, and hence there is a vector x_0 such that $p(A)x_0 = 0$ only for $p = 0$. Fix a polynomial p_0 with $Tx_0 = p_0(A)x_0$. We show $T = p_0(A)$. For any vector y there are polynomials r and s such that $Ty = r(A)y$ and $T(x_0 + y) = s(A)(x_0 + y)$. Then

$$(p_0(A) - s(A))x_0 = (s(A) - r(A))y;$$

call this vector z. If $z = 0$ then $p_0 = s$ and $s(A)y = r(A)y$, so that $Ty = p_0(A)y$. If $z \neq 0$, note first that $p(A)z = 0$ only for $p = 0$; since T commutes with A we get

$$Tz = p_0(A)z = r(A)z,$$

so that $p_0 = r$ and $Ty = p_0(A)y$ as before. Because y is arbitrary, $T = p_0(A)$ as asserted.

We conclude with a result suggested by P. Rosenthal. In his paper [**5**] with H. Radjavi, it is shown that in certain circumstances, if $\mathscr{A}$ is an algebra of bounded linear transformations on a Hilbert space such that $\mathscr{A}^*$ leaves invariant every closed invariant subspace of $\mathscr{A}$, then $\mathscr{A}$ is selfadjoint.

THEOREM 3. *If A is a bounded linear transformation on a Hilbert space such that $\mathscr{L}(A) \subset \mathscr{L}(A^*)$, then A^* is a polynomial in A. In particular, A is normal.*

PROOF. By Theorem 2 it is enough to show that $A^*Ax = AA^*x$ for every vector x. Let $M_x = \{p(A)x \mid p \text{ polynomial}\}$.

Assume first that M_x is finite-dimensional. The restrictions of A and A^* to M_x inherit the hypothesis; on choosing an orthonormal basis for M_x so that the matrix of $A|M_x$ is triangular, this is seen to imply that the matrix is actually diagonal, and hence that $A^*Ax=AA^*x$.

If M_x is infinite-dimensional, then $p(A)x=0$ implies $p=0$. By hypothesis there are polynomials r, s, and t such that $A^*x=r(A)x$, $A^*Ax=s(A)Ax$, and $A^*(Ax-\lambda x)=t(A)(Ax-\lambda x)$. Hence

$$(s(A)A - \lambda r(A))x = t(A)(A - \lambda)x,$$
$$zs(z) - \lambda r(z) = t(z)(z - \lambda),$$

and $s(\lambda)=r(\lambda)$ for all $\lambda\neq 0$. Consequently $s=r$ and $A^*Ax=s(A)Ax=r(A)Ax=AA^*x$ as required.

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Department of Mathematics, Dalhousie University, Halifax, Nova Scotia, Canada