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SOME REMARKS CONCERNING THE NUMBER THEORETIC FUNCTIONS $\omega(n)$ AND $\Omega(n)$

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ABSTRACT. In this paper we establish a duality relation between the number theoretic functions $\omega(n)$ and $\Omega(n)$, and we investigate some of its consequences, one of which concerns the Riemann zeta function.

1. **Introduction.** $\omega(n)$ is the number of distinct prime divisors of the positive integer n, and $\Omega(n)$ is the total number of prime divisors of n. That is, $\omega(1)=\Omega(1)=0$ and if $n=p_1^{e_1}\cdot\cdots\cdot p_r^{e_r}$ then $\omega(n)=r$ and $\Omega(n)=e_1+\cdots+e_r$. From their definitions it is not to be expected that there is much of a relation between ω and Ω. However, we will exhibit below a remarkable duality relation between the two functions and deduce some consequences of this relation.

2. **A duality between ω and Ω.** We begin by noting that $\omega(n)=O(\log n/\log\log n)$ [**1**, p. 355] from which it is easily seen that if $\operatorname{Re} s=\sigma>1$ then $\sum_{n=1}^{\infty}(z^{\omega(n)}/n^s)$ is an entire function of z. Also, for $\sigma>1$ and $|z|<2^\sigma$, it follows that

$$\sum_{n=1}^{\infty}\frac{z^{\Omega(n)}}{n^s}=\prod_p\left(1+\frac{z}{p^s}+\frac{z^2}{p^{2s}}+\cdots\right)$$
$$=\prod_p\left(\frac{1}{1-z/p^s}\right)$$

is an analytic function of z.

We now have

THEOREM 1. *If* $|z|<2^\sigma$ *and* $\sigma>1$ *then*

$$\left(\sum_{n=1}^{\infty}\frac{(1-z)^{\omega(n)}}{n^s}\right)\left(\sum_{n=1}^{\infty}\frac{z^{\Omega(n)}}{n^s}\right)=\zeta(s). \tag{1}$$

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PROOF.

$$\left(\sum_{n=1}^{\infty}\frac{(1-z)^{\omega(n)}}{n^s}\right)\left(\sum_{n=1}^{\infty}\frac{z^{\Omega(n)}}{n^s}\right)$$
$$=\prod_p\left(1+\frac{1-z}{p^s}+\frac{1-z}{p^{2s}}+\cdots\right)\prod_p\left(1+\frac{z}{p^s}+\frac{z^2}{p^{2s}}+\cdots\right)$$
$$=\prod_p\left(1+\frac{1-z}{p^s-1}\right)\left(\frac{p^s}{p^s-z}\right)=\prod_p\left(\frac{p^s-z}{p^s-1}\right)\left(\frac{p^s}{p^s-z}\right)=\zeta(s)$$

which completes the proof.

Now if, in addition to $|z|<2^\sigma$, we also require that $|1-z|<2^\sigma$, then by replacing z by $1-z$ in (1) we obtain

$$\text{(2)}\quad \zeta(s)=\left(\sum_{n=1}^{\infty}\frac{(1-z)^{\omega(n)}}{n^s}\right)\left(\sum_{n=1}^{\infty}\frac{z^{\Omega(n)}}{n^s}\right)=\left(\sum_{n=1}^{\infty}\frac{(1-z)^{\Omega(n)}}{n^s}\right)\left(\sum_{n=1}^{\infty}\frac{z^{\omega(n)}}{n^s}\right)$$

which is the announced duality relation.

REMARK. All formulas derived from (2) are dual in ω and Ω.

COROLLARY.

$$\text{(3)}\qquad \sum_{d|n}(1-z)^{\omega(d)}z^{\Omega(n/d)}=\sum_{d|n}(1-z)^{\Omega(d)}z^{\omega(n/d)}=1$$

for all n and all z.

PROOF. Equate coefficients in (2).

If we set $z=\frac{1}{2}$ in (3) we obtain

$$\text{(4)}\qquad \sum_{d|n}2^{-\omega(d)-\Omega(n/d)}=\sum_{d|n}2^{-\Omega(d)-\omega(n/d)}=1$$

for all n.

One can obtain other interesting nonobvious relations from (3) e.g., by differentiating termwise. Usually, relations involving elementary number theoretic functions admit a natural interpretation. However, we are unable to find any such natural interpretation for the above relations.

3. **Analytic continuation of $\sum_{n=1}^{\infty}(z^{\omega(n)}/n^s)$.** For any fixed z and $\operatorname{Re} s=\sigma>1$ we define $g_z(s)=\sum_{n=1}^{\infty}(z^{\omega(n)}/n^s)$ and consider the analytic continuation of $g_z(s)$ (as a function of s).

We first have

THEOREM 2. *For $|1-z|\leqq 2$ and $\sigma>1$, we have*

$$\text{(5)}\qquad \frac{g_z'(s)}{g_z(s)}=\frac{z\zeta'(s)}{\zeta(s)}+z(z-1)\sum_p\frac{\log p}{(p^s+z-1)(p^s-1)}.$$

PROOF. From (2) we have

$$g_z(s)\cdot\prod_p\left(1+\frac{1-z}{p^s}+\frac{(1-z)^2}{p^{2s}}+\cdots\right)=\zeta(s)$$

which can be rewritten as

$$g_z(s)=\zeta(s)\prod_p\left(\frac{p^s+z-1}{p^s}\right).$$

Taking logarithmic derivatives, we find

$$\begin{aligned}\frac{g_z'(s)}{g_z(s)}&=\frac{\zeta'(s)}{\zeta(s)}+\sum_p\left(\frac{p^s}{p^s+z-1}-1\right)\log p\\&=\frac{\zeta'(s)}{\zeta(s)}+\sum_p\left(\frac{1-z}{p^s+z-1}\right)\log p\\&=\frac{\zeta'(s)}{\zeta(s)}+\sum_p\left(\frac{1-z}{p^s+z-1}-\frac{1-z}{p^s-1}\right)\log p+\sum_p\frac{(1-z)\log p}{p^s-1}\\&=\frac{z\zeta'(s)}{\zeta(s)}+z(z-1)\sum_p\frac{\log p}{(p^s+z-1)(p^s-1)}\end{aligned}$$

which completes the proof.

REMARKS. If $|1-z|\leqq 2^{1/2}$, then

$$R_z(s)=\sum_p\frac{\log p}{(p^s+z-1)(p^s-1)}$$

is regular for $\sigma>\frac{1}{2}$. It then follows that $g_z'(s)/g_z(s)$ is regular for $\sigma>\frac{1}{2}$ except at $s=1$ and at the (possible) zeros of $\zeta(s)$ situated at the right of the line $\sigma=\frac{1}{2}$. Thus, (5) can be rewritten as

$$g_z'(s)/g_z(s)=z\zeta'(s)/\zeta(s)+z(z-1)R_z(s).$$

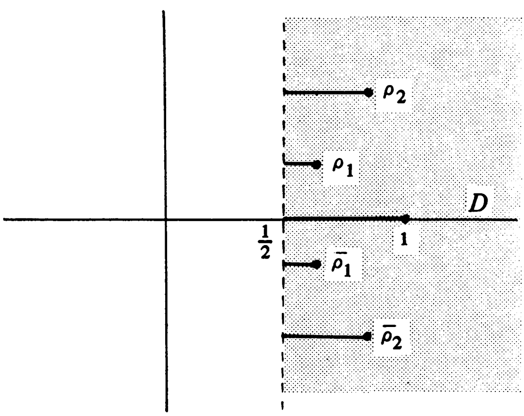

FIGURE 1

After integration we have $g_z(s)=\zeta^z(s)\exp(P_z(s))$ where $P_z(s)$ is regular for $\sigma>\frac{1}{2}$. Thus, $g_z(s)$ is regular in the region D below, where the ρ's stand for the (possible) zeros of $\zeta(s)$ which lie at the right of $\sigma=\frac{1}{2}$.

Finally (assuming that all zeros of the zeta function are simple) we note that, surprisingly enough, regardless of which z is chosen subject to $0<|1-z|\leqq 2^{1/2}$ and $z\neq 2$, the set of singularities of $g_z(s)$ in the half-plane $\sigma>\frac{1}{2}$ always consists of the same points, namely $s=1$ and the (possible) zeros of the zeta function lying to the right of $\sigma=\frac{1}{2}$. This seems to lend credence to the Riemann Hypothesis.

Reference

1. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 4th ed., Oxford Univ. Press, London, 1960.

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