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THE STRICT TOPOLOGY FOR P-SPACES

ROBERT F. WHEELER

ABSTRACT. A P-space is a completely regular Hausdorff space X in which every G_δ is open. It is shown that the generalized strict topologies β and β_0 coincide on $C^*(X)$, and that strong measure-theoretic properties hold; in particular, $(C^*(X), \beta)$ is always a strong Mackey space. As an application, an example is constructed of a non-quasi-complete locally convex space in which closed totally bounded sets are compact.

1. **Introduction.** All topological spaces considered here are assumed to be completely regular Hausdorff. If X is such a space, then $C^*(X)$ denotes the space of bounded real-valued continuous functions on X. The strict topology β on $C^*(X)$ was introduced by Buck **[6]** for locally compact X; combining his ideas with the measure-theoretic concepts of Varadarajan **[18]**, Sentilles **[17]** considered locally convex topologies β_0, β, and β_1 on $C^*(X)$, X completely regular, which yield the spaces $M_t(X)$, $M_\tau(X)$, and $M_\sigma(X)$ of tight, τ-additive, and σ-additive Baire measures as duals. A useful reference for the notions of uniform σ- and τ-additivity is **[10]**. We assume the results of these papers as needed.

The principal source of information about P-spaces is the Gillman-Jerison text **[11]**. Nondiscrete P-spaces are remote from the locally compact spaces studied in measure theory; indeed every compact subset of such a space is finite. Nevertheless, it is shown here that certain aspects of the strict topology in the locally compact case remain valid for P-spaces. For example, the topologies β and β_0 coincide, and $(C^*(X), \beta)$ is always a strong Mackey space. The latter result can be obtained using either the well-known techniques of Conway **[8]** for paracompact locally compact spaces or via the more recent ideas of uniform σ- and τ-additivity, giving an opportunity to compare and contrast these methods. Every discrete space is a P-space, and some results of this paper can be viewed as extensions of the work of Collins **[7]** in the discrete case. On the other

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hand, the existence of a P-space which admits no complete uniform structure [**11**, 9L] permits the construction of counterexamples to two recent conjectures in functional analysis and topological measure theory.

The following properties of P-spaces will be needed: every zero-set of a real-valued continuous function is open (indeed the Baire sets of a P-space are precisely the clopen subsets). If (f_n) is a uniformly bounded sequence of functions in $C^*(X)$, then $f(x)=\sup f_n(x)$, and $g(x)=\inf f_n(x)$ are also members of $C^*(X)$. The latter property resembles (but is stronger than) the requirement that $C^*(X)$ with the usual ordering be a conditionally σ-complete lattice.

2. **Properties of β and β_0 for P-spaces.** For any space X, let X_d denote the underlying set of X, endowed with the discrete topology. Then $C^*(X)$ can be considered as a subspace of $C^*(X_d)$.

THEOREM 2.1. *If X is a P-space, then* (a) $(C^*(X), \beta_0)$ *is topologically isomorphic under the inclusion map to a subspace of* $(C^*(X_d), \beta_0)$, *and the latter space is its completion*; (b) $(C^*(X), \beta_0)$ *is sequentially complete*; (c) $M_\tau(X)=M_t(X)=l^1(X)$.

PROOF. Since every compact subset of X is finite, the associated k-space of X is X_d; now (a) follows from Theorem 6 of [**10**]. The result of (b) is an easy consequence of the fact that, in a P-space, the pointwise limit of any uniformly bounded sequence of continuous functions is again continuous. The final result was proved by Babiker [**2**].

Now we use the Conway-LeCam technique to show that β and β_0 are identical in our setting.

THEOREM 2.2. *If X is a P-space, then $(C^*(X), \beta_0)$ is a strong Mackey space, and $\beta=\beta_0$.*

PROOF. Since β and β_0 yield the same dual space (2.1(c)), and always $\beta_0\leqq\beta$, the first result implies the second. Let A be a subset of $M_t(X)$ such that every sequence in A has a weak*-cluster point (in $M_t(X)$). If A is not uniformly tight, then we can find $\varepsilon_0>0$, pairwise disjoint compact (equivalently, finite) subsets (D_n) of X and members (μ_n) of A with

$$|\mu_n|\left(X\backslash\bigcup_{i=1}^{n-1} D_i\right) > \varepsilon_0 \tag{1}$$

and

$$|\mu_n|\left(X\backslash\bigcup_{i=1}^{n} D_i\right) < \varepsilon_0/4\forall n. \tag{2}$$

Since every G_δ in X is open, every pairwise disjoint sequence of closed sets in X is discrete. Applying this to the sequence of compact sets (D_n), we can easily obtain a sequence (F_n) of pairwise disjoint closed sets with each

D_n contained in the interior of F_n. Then (F_n) is also a discrete sequence of sets. Let $D_n=\{x_{i,n}:1\leqq i\leqq i_n\}$, and for each n, choose $f_n\in C^*(X)$ with $f_n(x_{i,n})=\operatorname{sgn}\mu_n(\{x_{i,n}\})$, $f_n|X\backslash F_n\equiv 0$, and $\|f_n\|\leqq 1$. Then the map $T:l^\infty\to C^*(X)$ defined by $T(\alpha)=\sum_{n=1}^\infty \alpha_n f_n$ is $\beta_0-\beta_0$ continuous. Thus the adjoint T^* maps $M_t(X)$ into l^1, and T^*A is relatively $\sigma(l^1, l^\infty)$-countably compact, hence relatively norm compact. Hence there is a positive integer n_0 such that $|T^*\mu_n(e_n)|<\varepsilon_0/2\,\forall n\geqq n_0$, where e_n is the nth unit vector. However, it is easily verified that $|T^*\mu_n(e_n)|=|\mu_n(f_n)|>\varepsilon_0/2$, and we have a contradiction.

We now give a characterization of P-spaces and discrete spaces X in terms of behavior of $B=\{f\in C^*(X):\|f\|\leqq 1\}$.

THEOREM 2.3. *Let X be completely regular Hausdorff. Then*

(a) *B is β- (or β_0-) compact if and only if X is discrete;*

(b) *B is β- (or β_0-) countably compact if and only if X is a P-space;*

(c) *B is β_0-totally bounded if and only if every compact subset of X is finite.*

PROOF. (a) If B is β_0-compact and $p\in X$, let (f_α) be a net in B which converges pointwise to the characteristic function of $\{p\}$. Then any β_0-cluster point of (f_α) necessarily coincides with this characteristic function, so $\{p\}$ is open and X is discrete. Conversely if X is discrete, then B is β-totally bounded [7, Theorem 4.1] and β-complete, hence β-compact. (b) If B is β_0-countably compact, then, replacing $\{p\}$ and (f_α) in (a) by a zero-set Z and sequence (f_n), we find X to be a P-space (every zero-set is open). Conversely, if X is a P-space and (f_n) is a sequence in B, define an equivalence relation on X by $x\sim y$ iff $f_n(x)=f_n(y)\forall n$. Each equivalence class is a zero-set, hence open, so the quotient space Y is discrete. Let $\pi:X\to Y$ be the quotient map; there is a sequence (g_n) in the unit ball of $C^*(Y)$ with $g_n\circ\pi=f_n\forall n$. From (a), (g_n) has a β_0-cluster point g_0, and $g_0\circ\pi$ is a β_0-cluster point of (f_n). Since $\beta_0=\beta$ for X a P-space, the result follows. (c) Since β_0 and the compact-open topology agree on B, this result is immediate.

Now we have that X a P-space$\Rightarrow B$ is β-totally bounded$\Rightarrow$compact subsets of X are finite. However, neither implication can be reversed as we now show. Varadarajan [**18**, pp. 225–227] discusses two spaces which may be described as follows: Let N denote the set of positive integers, and let $\mathscr{F}$ be the filter of subsets of N which have density one [**11**, 6U], with p a fixed cluster point of $\mathscr{F}$ in βN. Let $\mathscr{U}$ be the unique ultrafilter on N which refines $\mathscr{F}$ and converges to p. Then $N\cup\{p\}$ can be topologized by requiring each point of N to be open and neighborhoods of p to be of the form $\{p\}\cup F$, where $F\in\mathscr{F}$ or $F\in\mathscr{U}$; call the resulting spaces V and E (E has the relative topology of βN). It can be shown (using, for example, Theorem 2.13 of [15]) that $\beta=\beta_0$ on $C^*(E)$. Since compact subsets of E are finite, $B\subset C^*(E)$ is then β-totally bounded, but E is not a P-space.

On the other hand, compact subsets of V are finite, and β and β_0 yield the same dual space for $C^*(V)$ [**15**, Proposition 3.4], yet $B\subset C^*(V)$ is not β-totally bounded. If it were, then, by a duality result of Grothendieck [**14**, p. 266], each β-equicontinuous subset of $M_\tau(V)$ would be norm-totally bounded. For each $x\in V$, let $\delta(x)$ be the point mass at x; define $\mu_n=n^{-1}(\sum_{i=1}^n \delta(i))$, $\mu_0=\delta(p)$. Then (μ_n) is weak*-convergent to μ_0 in $M_\tau^+(V)$ [**18**, p. 226] and so $A=\{\mu_n:n\geqq 0\}$ is β-equicontinuous [**17**, Theorem 5.2], but, as is easily seen, not norm-totally bounded.

3. **Measure-theoretic properties of P-spaces.** Any space X for which $M_\sigma(X)=M_\tau(X)$ is realcompact; Babiker [**2**], [**3**] has shown that the converse is true for P-spaces (with certain cardinality assumptions). By minor modifications of his arguments it can be shown that (in the terminology of [**19**]) a P-space satisfies $M_\tau=M_s$ if and only if it is topologically complete (no cardinality assumptions needed). However, a topologically complete P-space need not be paracompact, or even normal [**1**].

The next result shows that P-spaces have the following curious property: any set of τ-additive measures which is well-behaved with respect to sequences (uniformly σ-additive) is necessarily well-behaved with respect to nets (uniformly τ-additive). This is true in spite of the fact that $M_\sigma(X)\neq M_\tau(X)$ for certain P-spaces X (§4). In the following, let ξ denote the family of uniformly bounded equicontinuous subsets of $C^*(X)$, and let $\mathscr{T}(\xi)$ denote the topology on $M_\sigma(X)$ of uniform convergence on members of ξ. Since sequences in $C^*(X)$ which are either norm convergent to 0 or monotone decreasing and pointwise convergent to 0 are equicontinuous, it is easy to see that $(M_\sigma(X), \mathscr{T}(\xi))$ is a complete locally convex space.

THEOREM 3.1. *If X is a P-space, then the following conditions on a subset H of $M_\tau(X)$ are equivalent*:

(a) *uniformly τ-additive*;
(b) *relatively weak*-compact in $M_\tau(X)$*;
(c) *every sequence in H has a weak*-cluster point in $M_\tau(X)$*;
(d) *every sequence in H has a weak*-cluster point in $M_\sigma(X)$*;
(e) *uniformly σ-additive*;
(f) *norm-totally bounded*;
(g) *$\mathscr{T}(\xi)$-totally bounded.*

PROOF. The implications (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d)$\Leftrightarrow$(e) and (f)$\Rightarrow$(g)$\Rightarrow$(d) are true for any X ((d)$\Leftrightarrow$(e) was shown by Varadarajan [**18**, p. 203], and (g)$\Rightarrow$(d) follows from $\mathscr{T}(\xi)$-completeness of $M_\sigma(X)$). For X a P-space, we show (e)$\Rightarrow$(a)$\Rightarrow$(f). If H is uniformly σ-additive, then so are $H^+=\{\mu^+:\mu\in H\}$ and $H^-=\{\mu^-:\mu\in H\}$ [**10**, p. 120]; hence we may as well assume that H consists of nonnegative measures. If H is not uniformly τ-additive, there is a net $(f_\alpha)\downarrow 0$ in $C^*(X)$ and $\varepsilon_0>0$ such that

$$\sup\{\mu(f_\alpha):\mu\in H\} > \varepsilon_0 \forall\alpha.$$

Using the τ-additivity of each member of H, we can find a sequence $\alpha_1>\alpha_2>\cdots$ of indices and members (μ_n) of H such that $\mu_n(f_{\alpha_n})>\varepsilon_0$ but $\mu_n(f_{\alpha_{n+1}})<\varepsilon_0/2\forall n$. Let $f_0=\inf f_{\alpha_n}$. Then $f_0\in C^*(X)$ and $(f_{\alpha_n}-f_0)\downarrow 0$, so $\mu(f_{\alpha_n}-f_0)\to 0$ uniformly with respect to $\mu\in A$. This implies the existence of an integer n_0 such that $\mu_{\alpha_n}(f_{\alpha_n}-f_{\alpha_{n+1}})<\varepsilon_0/4$ for $n\geqq n_0$, a contradiction.

Since the uniformly τ-additive sets are precisely the β-equicontinuous sets, (a)$\Rightarrow$(f) follows from 2.3 and the Grothendieck result mentioned in the previous section.

Note that the equivalence of (a) and (c) furnishes an alternate proof that $(C^*(X),\beta)$ is a strong Mackey space, independent of 2.2.

COROLLARY 3.2. *If X is a P-space, then* (a) *the norm and weak* topologies agree on β-equicontinuous subsets of $M_\tau(X)$*; (b) *$M_\tau(X)$ is weak*-sequentially complete*; *in fact, every weak*-Cauchy sequence in $M_\tau(X)$ is norm-convergent.*

PROOF. Since $M_\tau(X)$ is a norm-closed subspace of $M(X)$, the Banach dual of $C^*(X)$, the first assertion follows from the equivalence of (a) and (f) in 3.1. It is known [**18**, p. 195] that $M_\sigma(X)$ is weak*-sequentially complete. Thus a weak*-Cauchy sequence in $M_\tau(X)$ satisfies 3.1(d), so it is β-equicontinuous, and the result follows.

As an immediate consequence of 3.2(a), the finest topology on the dual of $(C^*(X),\beta)$ which agrees with the weak* topology on β-equicontinuous sets is the norm topology. The corresponding result for discrete X was proved by Collins [**7**, Theorem 4.1]. Note that any weak*-convergent sequence in $M(X)$ is weakly convergent; since βX is an F-space when X is a P-space, this follows from a result of Seever [**16**].

4. **A counterexample.** An example of a non-realcompact P-space S is recorded in [**11**, 9L]. Applying Shirota's theorem [**11**, p. 229] and 2.2, we have: $(C^*(S),\beta)$ is a strong Mackey space, although S admits no complete uniform structure. This resolves negatively a conjecture advanced by the author in [**19**]. The conjecture, however, remains open for the category of k-spaces (in particular, for locally compact spaces).

The class of locally convex spaces in which closed totally bounded sets are compact has been examined in [**4**] and [**9**]. Answering a question posed by Buchwalter [**5**], Haydon [**13**] has given an example of a non-quasi-complete space with this property. The example offered here, obtained independently by the author, is of a very different sort.

EXAMPLE 4.1. $(M_\tau(S), \mathscr{T}(\xi))$ is a non-quasi-complete locally convex space in which closed totally bounded sets are compact.

Note that $M_\tau(S)$ is identical as a vector space to $l^1(S_d)$, where S_d has cardinal $\aleph_2$. The fact that closed totally bounded sets are compact follows readily from 3.1. On the other hand, there is a natural embedding $j: S \to (M_\tau(S), \mathscr{T}(\xi))$. Arguing as in [**19**], it can be shown that $j(S)$ is $\mathscr{T}(\xi)$-closed and bounded in $M_\tau(S)$, yet the $\mathscr{T}(\xi)$-closure of $j(S)$ in $M_\sigma(S)$ is (a copy of) the Hewitt real-compactification of S, hence properly contains $j(S)$. Thus $j(S)$ is not $\mathscr{T}(\xi)$-complete.

5. **Possible extensions.** The results obtained here for P-spaces are not valid, in general, for other classes of highly disconnected spaces. For example, assume the continuum hypothesis, and let p be a P-point of $\beta N \backslash N$ [**11**, 6V]. Then $X \doteq \beta N \backslash \{p\}$ is extremally disconnected and locally compact. However, $(C^*(X), \beta)$ is not a Mackey space; the argument is similar to that of Conway for the ordinals less than ω_1 [**8**, p. 481]. But under the additional assumption that compact subsets of X are finite, strong results for highly disconnected spaces have been obtained recently by Haydon [**12**].

REFERENCES

1. K. Alster and R. Engelking, *Subparacompactness and product spaces*, Bull. Acad. Polon. Sci. **20** (1972), 763–767.

2. A. G. A. Babiker, *On almost discrete spaces*, Mathematika **18** (1971), 163–167.

3. ———, *Uniform continuity of measures on completely regular spaces*, J. London Math. Soc. **5** (1972), 451–458.

4. K. Brauner, Doctoral Dissertation, Univ. of Calif., Berkeley, 1972.

5. H. Buchwalter, *Fonctions continues et mesures sur un espace complétement régulier*, Brussels Conference, 1972 (preprint).

6. R. C. Buck, *Bounded continuous functions on a locally compact space*, Michigan Math. J. **5** (1958), 95–104. MR **21** #4350.

7. H. S. Collins, *On the space $l^\infty(S)$, with the strict topology*, Math. Z. **106** (1968), 361–373. MR **39** #763.

8. J. B. Conway, *The strict topology and compactness in the space of measures.* II, Trans. Amer. Math. Soc. **126** (1967), 474–486. MR **34** #6503.

9. J. Dazord and M. Jourlin, *Sur les precompacts d'un espace localement convexe*, C. R. Acad. Sci. Paris **274** (1972), 463–466, 550–553.

10. D. H. Fremlin, D. J. H. Garling, and R. G. Haydon, *Bounded measures on topological spaces*, Proc. London Math. Soc. **25** (1972), 115–136.

11. L. Gillman and M. Jerison, *Rings of continuous functions*, University Series in Higher Math., Van Nostrand, Princeton, N.J., 1960. MR **22** #6994.

12. R. Haydon, *On compactness in spaces of measures and measurecompact spaces* (preprint).

13. R. Haydon,, *Sur un problème de H. Buchwalter*, C. R. Acad. Sci. Paris **275** (1972), 1077–1080.

14. G. Köthe. *Topologische linear Räume*. I, Die Grundlehren der math. Wissenschaften, Band 107, Springer-Verlag, Berlin, 1960; English transl., Die Grundlehren der math. Wissenschaften, Band 159, Springer-Verlag, New York, 1969. MR **24** #A411; **40** #1750.

15. S. E. Mosiman and R. F. Wheeler, *The strict topology in a completely regular setting: relations to topological measure theory*, Canad. J. Math. **24** (1972), 873–890.

16. G. Seever, *Measures on F-spaces*, Trans. Amer. Math. Soc. **133** (1968), 267–280. MR **37** #1976.

17. F. D. Sentilles, *Bounded continuous functions on a completely regular space*, Trans. Amer. Math. Soc. **168** (1972), 311–336. MR **45** #4133.

18. V. S. Varadarajan, *Measures on topological spaces*, Mat. Sb. **55 (97)** (1961), 35–100; English transl., Amer. Math. Soc. Transl. (2) **48** (1965), 161–228. MR **26** #6342.

19. R. F. Wheeler, *The strict topology, separable measures, and paracompactness*, Pacific J. Math. (to appear).

DEPARTMENT OF MATHEMATICS, NORTHERN ILLINOIS UNIVERSITY, DEKALB, ILLINOIS 60115