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PERFECT TORSION THEORIES

PAUL E. BLAND

ABSTRACT. The purpose of this paper is to introduce the study of perfect torsion theories on Mod R and to dualize the concept of divisible module. A torsion theory on Mod R is called perfect if every torsion module has a projective cover. It is shown for such a theory that the class of torsion modules is closed under projective covers if and only if the class of torsion free modules is closed under factor modules. In addition, it is shown that this condition on a perfect torsion theory is equivalent to its idempotent radical being an epiradical. Codivisible covers of modules are also introduced and we are able to show that any module which has a projective cover has a codivisible cover. Codivisible covers are then characterized in terms of the projective cover of the module and the torsion submodule of the kernel of the minimal epimorphism.

1. **Preliminaries.** Throughout this paper R will denote an associative ring with identity and our attention will be confined to the category Mod R of unital right R-modules. If M and N are modules in Mod R, then $\mathrm{Hom}_R(M, N)$ will be abbreviated by $[M, N]$.

In [2], Dickson defined a torsion theory on Mod R to be a pair $(\mathscr{A}, \mathscr{B})$ of classes of modules such that:

(a) $\mathscr{A} \cap \mathscr{B} = \{0\}$.

(b) If $A \to A^* \to 0$ is exact with $A \in \mathscr{A}$, then $A^* \in \mathscr{A}$.

(c) If $0 \to B^* \to B$ is exact with $B \in \mathscr{B}$, then $B^* \in \mathscr{B}$.

(d) For each module M in Mod R there is an exact sequence $0 \to A \to M \to B \to 0$ with $A \in \mathscr{A}$ and $B \in \mathscr{B}$.

Modules in $\mathscr{A}$ are called torsion and those in $\mathscr{B}$ torsion free. If the operators l and r are defined on a class $\mathscr{C}$ of modules by

$$l(\mathscr{C}) = \{M \in \mathrm{Mod}\, R \mid [M, C] = 0 \text{ for all } C \in \mathscr{C}\} \quad \text{and}$$
$$r(\mathscr{C}) = \{M \in \mathrm{Mod}\, R \mid [C, M] = 0 \text{ for all } C \in \mathscr{C}\},$$

then $(\mathscr{A}, \mathscr{B})$ is a torsion theory if and only if $r(\mathscr{A}) = \mathscr{B}$ and $l(\mathscr{B}) = \mathscr{A}$ [2, p. 229, Proposition 3.3]. Furthermore, if $(\mathscr{A}, \mathscr{B})$ is a torsion theory,

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then $\mathscr{A}$ is closed under isomorphic images, factor modules, extensions and direct sums while $\mathscr{B}$ is closed under isomorphic images, submodules, extensions and direct products [**2**, p. 226, Theorem 2.3]. By saying that a class $\mathscr{C}$ of modules is closed under extensions we mean that $M \in \mathscr{C}$ whenever N is a submodule of M such that N and M/N are in $\mathscr{C}$. If $(\mathscr{A}, \mathscr{B})$ is a torsion theory such that $\mathscr{B}$ is closed under injective hulls [**3**], then $(\mathscr{A}, \mathscr{B})$ is called hereditary. It is known that $(\mathscr{A}, \mathscr{B})$ is hereditary if and only if $\mathscr{A}$ is closed under submodules [**2**, Theorem 2.9, p. 227]. If each element of $\mathscr{A}$ has a projective cover, then $(\mathscr{A}, \mathscr{B})$ will be called (right) perfect. Clearly, every torsion theory on Mod R is perfect if and only if R is a (right) perfect ring. (See [**1**] for a characterization of right perfect rings.) A projective module P is a projective cover of a module M if there exists an epimorphism $\pi: P \to M$ with coessential kernel. $P(M)$ will be reserved to denote the projective cover of a module M whenever such exists. A submodule K of M is said to be coessential in M if $N=M$ whenever N is a submodule of M such that $K+N=M$. We will call any epimorphism with a coessential kernel minimal. If $(\mathscr{A}, \mathscr{B})$ is a perfect torsion theory such that $\mathscr{A}$ is closed under projective covers, then $(\mathscr{A}, \mathscr{B})$ will be referred to as cohereditary.

An object functor T: Mod $R \to$ Mod R is said to be a radical if $T(M) \subseteq M$, $f: M \to N$ implies that $f(T(M)) \subseteq T(N)$ and $T(M/T(M))=0$. If $f: M \to N$ is any epimorphism, then a radical T, such that $f(T(M))=T(N)$, is an epiradical. A radical T, such that $T(T(M))=T(M)$ for all modules M, is called idempotent. It is well known that $(\mathscr{A}, \mathscr{B})$ is a torsion theory on Mod R if and only if there exists an idempotent radical T on Mod R such that

$$\mathscr{A} = \{A \in \text{Mod } R \mid T(A) = A\} \quad \text{and} \quad \mathscr{B} = \{B \in \text{Mod } R \mid T(B) = 0\}$$

[**5**, p. 2, Proposition 0.1]. Furthermore, this correspondence is one-to-one [**7**, p. 6, Proposition 2.3]. If $(\mathscr{A}, \mathscr{B})$ is a torsion theory with idempotent radical T, then $(\mathscr{A}, \mathscr{B})$ is hereditary if and only if T is a left exact functor [**7**, Proposition 2.6, p. 8]. (In this context if $f: M \to N$, then $T(f)$ is the restriction of f to $T(M)$.)

2. Cohereditary torsion theories.

THEOREM 2.1. *If $(\mathscr{A}, \mathscr{B})$ is a perfect torsion theory, then $(\mathscr{A}, \mathscr{B})$ is cohereditary if and only if $\mathscr{B}$ is closed under factor modules.*

PROOF. Let $A \in \mathscr{A}$ and suppose that $\mathscr{B}$ is closed under factor modules. If B is any member of $\mathscr{B}$ and $f \in [P(A), B]$, set $g=\eta \circ f$ where $\eta: B \to B/f(K)$ is the canonical epimorphism and K is the kernel of the minimal epimorphism $\pi: P(A) \to A$. Since $g(K)=0$, we have an induced mapping

$g^*:A \to B/f(K)$ which must be the zero map since $B/f(K) \in \mathscr{B}$. Consequently, if $x \in P(A)$, then $0 = g^* \circ \pi(x) = g(x) = f(x) + f(K)$ and so $f(x) \in f(K)$. Hence if $z = x - k$ where $k \in K$ is such that $f(k) = f(x)$, then $z \in \ker f$. Thus $x \in \ker f \dotplus K$ and so $P(A) = \ker f + K$. Now K is coessential in $P(A)$ and therefore $P(A) = \ker f$. Hence $P(A) \in \mathscr{A}$ and so $(\mathscr{A}, \mathscr{B})$ is cohereditary.

The converse follows easily by using the projectivity of projective covers.

The following lemma will be useful.

Lemma 2.2. *Let $\mathscr{C}$ be a nonempty class of modules each of which has a projective cover. If $S(C, M) = \sum f(P(C))$ $(f \in [P(C), M])$ for any module M and any $C \in \mathscr{C}$, then $T(M) = \sum S(C, M)$ $(C \in \mathscr{C})$ defines an idempotent epiradical on* Mod R.

Proof. The fact that T is an idempotent radical follows in a straightforward fashion from the definition of T. Now suppose that $g: M \to N$ is an epimorphism and let $y \in T(N)$. If $y = \sum f(z)$ where $f \in [P(C), N]$ for some $C \in \mathscr{C}$, then for each f there is a mapping $h \in [P(C), M]$ such that $f = g \circ h$. Thus if $x = \sum h(z)$, then $x \in T(M)$ and $g(x) = \sum g \circ h(z) = \sum f(z) = y$. Hence $g(T(M)) = T(N)$ and so T is an epiradical.

The following theorem establishes a one-to-one correspondence between perfect torsion theories which are cohereditary and idempotent epiradicals. We need the following notation. If $\mathscr{C}$ is a nonempty class of modules, let $l^*(\mathscr{C}) = \{M \in \text{Mod } R \mid P(M) \text{ exists and } [P(M), C] = 0 \text{ for all } C \in \mathscr{C}\}$.

Theorem 2.3. *If $\mathscr{A}$ and $\mathscr{B}$ are nonempty classes of modules, then the following are equivalent*:

(a) *$(\mathscr{A}, \mathscr{B})$ is a cohereditary torsion theory.*

(b) *$r(\mathscr{A}) = \mathscr{B}$ and $l^*(\mathscr{B}) = l(\mathscr{B}) = \mathscr{A}$.*

(c) *There exists an idempotent epiradical T on* Mod R *such that $\mathscr{A} = \{A \in \text{Mod } R \mid T(A) = A\}$ and $\mathscr{B} = \{B \in \text{Mod } R \mid T(B) = 0\}$. Furthermore, each element of $\mathscr{A}$ has a projective cover.*

Proof. (b)$\Rightarrow$(c). Since $l^*(\mathscr{B}) = \mathscr{A}$ each element of $\mathscr{A}$ has a projective cover. Hence let T be the idempotent epiradical of Lemma 2.2 where the class $\mathscr{C}$ is replaced by the class $\mathscr{A}$. Now suppose that $A^* = T(A^*)$. Then for each $a \in A^*$ we have $a = \sum f_x(x)$ where each $f_x \in [P(A), A^*]$ for some $A \in \mathscr{A}$. Suppose next that we choose our notation so that $x \in P(A_x)$. Then if $S(A_a) = \sum \oplus P(A_x)$, then the R-linear mapping $f_a: S(A_a) \to A^*: \sum \oplus y \to \sum f_x(y)$ is such that $f_a(\sum \oplus x) = a$. Now $\mathscr{A}$ is closed under direct sums and it follows from $l^*(\mathscr{B}) = \mathscr{A}$ that $\mathscr{A}$ is closed under projective covers. Hence $S(A_a) \in \mathscr{A}$ and so $\sum \oplus S(A_a)$ $(a \in A^*)$ is a member of $\mathscr{A}$. Thus the R-linear mapping $\sum \oplus S(A_a) \to A^*: \sum \oplus z \to \sum f_a(z)$ is an epimorphism. Therefore A^* is an epimorph of an element in $\mathscr{A}$ and so $A^* \in \mathscr{A}$. Hence $\{A \in \text{Mod } R \mid T(A)=A\} \subseteq \mathscr{A}$. Conversely, if $A^* \in \mathscr{A}$, then $A^*=S(A^*, A^*) \subseteq \sum S(A, A^*)=T(A^*)$. Consequently, $T(A^*)=A^*$ and so $\mathscr{A} \subseteq \{A \in \text{Mod } R \mid T(A)=A\}$.

There is no difficulty in showing that $\mathscr{B}=\{B \in \text{Mod } R \mid T(B)=0\}$.

(c)$\Rightarrow$(a). It is immediate that (c) implies that $(\mathscr{A}, \mathscr{B})$ is a perfect torsion theory. Hence it remains only to show that $\mathscr{A}$ is closed under projective covers. Let $A \in \mathscr{A}$ and suppose that K is the kernel of the minimal epimorphism $\pi: P(A) \to A$. Since T is an epiradical, then the restriction μ of π to $T(P(A))$ is an epimorphism. Now suppose that f is the completing homomorphism for the diagram

$$\begin{array}{ccccc} & & P(A) & & \\ & & \downarrow{\scriptstyle \pi} & & \\ T(P(A)) & \xrightarrow{\mu} & A & \longrightarrow & 0 \end{array}$$

If $x \in P(A)$ and $k=x-f(x)$, then $\pi(k)=\pi(x)-\mu \circ f(x)=0$ and so $k \in K$. Hence $x \in T(P(A))+K$ and therefore $P(A)=T(P(A))+K$. But K is coessential in $P(A)$ and so $T(P(A))=P(A)$.

(a)$\Rightarrow$(b) is an easy exercise.

COROLLARY 2.4. *If $(\mathscr{A}, \mathscr{B})$ is a perfect torsion theory with idempotent radical T, then $(\mathscr{A}, \mathscr{B})$ is cohereditary if and only if T is an epiradical.*

It is now immediate that if $(\mathscr{A}, \mathscr{B})$ is a perfect torsion theory with idempotent radical T, then $(\mathscr{A}, \mathscr{B})$ is a hereditary, cohereditary torsion theory if and only if T is an exact functor.

3. **Codivisible modules.** The following definition dualizes the concept of a divisible module given by Lambek in [5, p. 8]. A module M is codivisible with respect to a torsion theory if every diagram of the form

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow & & \\ L & \xrightarrow{f} & N & \longrightarrow & 0 \end{array}$$

can be completed where $\ker f$ is torsion free. The usual argument now shows that a direct sum of modules is codivisible if and only if each factor is codivisible. Any homomorphism with torsion free kernel will be called free. In the discussion which follows we confine our attention to a given torsion theory with idempotent radical T.

LEMMA 3.1. *If $f: L \to M$ is a minimal, free epimorphism, then f is an isomorphism whenever M is codivisible.*

PROOF. If M is codivisible, then the diagram

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow{\scriptstyle 1_M} & & \\ L & \xrightarrow{f} & M & \longrightarrow & 0 \end{array}$$

can be completed by a homomorphism g. Hence $L = \ker f \oplus \operatorname{Im} g$ and so $\ker f = 0$ since $\ker f$ is coessential in L.

A codivisible module $C(M)$ is said to be a codivisible cover of a module M, if there exists a minimal, free epimorphism $\mu : C(M) \to M$. $C(M)$ will denote such a cover whenever it can be shown to exist.

THEOREM 3.2. *The codivisible cover (when it exists) is unique up to an isomorphism.*

PROOF. Let $\mu : C(M) \to M$ and $\mu^* : C(M)^* \to M$ be codivisible covers of M. Then the diagram

$$\begin{array}{ccccc} & & C(M) & & \\ & & \downarrow{\scriptstyle \mu} & & \\ C(M)^* & \xrightarrow{\mu^*} & M & \longrightarrow & 0 \end{array}$$

can be completed with a homomorphism g since μ^* is free. Hence it follows that $C(M)^* = \ker \mu^* + \operatorname{Im} g$. Thus g is an epimorphism since $\ker \mu^*$ is coessential in $C(M)^*$. Now $\ker g \subseteq \ker \mu$ and so it follows that g is a minimal, free epimorphism. Consequently, by Lemma 3.1, g must be an isomorphism.

We will now show that codivisible covers exist when projective covers exist.

THEOREM 3.3. *If $\pi : P(M) \to M$ is a projective cover of M, then $\mu : P(M)/T(\ker \pi) \to M$ is a codivisible cover of M where μ is the mapping induced by π.*

PROOF. Since $\ker \mu \cong \ker \pi / T(\ker \pi)$, then $\ker \mu$ is torsion free. Note also that $\ker \mu$ is coessential since the epimorph of a coessential module is coessential [6, Hilfssatz 3.1, p. 189]. Now consider the diagram

$$\begin{array}{ccccc} & & P(M) & & \\ & & \downarrow{\scriptstyle \text{nat}} & & \\ & {\scriptstyle g}\swarrow & P(M)/T(\ker \pi) & & \\ & & \downarrow{\scriptstyle h} & & \\ L & \xrightarrow{f} & N & \longrightarrow & 0 \end{array}$$

where f is free and g is the completing homomorphism given by the projectivity of $P(M)$. Since $g: T(\ker \pi) \to \ker f$ we see that $g(T(\ker \pi)) \subseteq T(\ker f) = 0$. Hence there is an induced mapping $g^*: P(M)/T(\ker \pi) \to L$ such that $f \circ g^* = h$. Thus $P(M)/T(\ker \pi)$ is codivisible and this completes the proof.

The above theorem shows that every module in Mod R has a codivisible cover when R is a perfect ring. It would be interesting to know under what conditions the universal existence of codivisible covers implies that of projective covers.

THEOREM 3.4. *If $\pi: P(M) \to M$ is a projective cover of M, then M is codivisible if and only if* $\ker \pi$ *is torsion.*

PROOF. Let $\mu: C(M) \to M$ be a codivisible cover of M. If M is codivisible then μ must be an isomorphism because of Lemma 3.1. But $\ker \mu \cong \ker \pi / T(\ker \pi)$ and so $T(\ker \pi) = \ker \pi$. Conversely, if $\ker \pi$ is torsion, then M is codivisible by Theorem 3.3.

THEOREM 3.5. *If $\mu: C(M) \to M$ is a codivisible cover of M, then $C(M)$ is torsionfree whenever M is torsionfree.*

PROOF. If M is torsionfree, then $0 \to \ker \mu \to C(M) \to M \to 0$ is exact with both $\ker \mu$ and M torsionfree. Hence $C(M)$ is torsionfree since the class of torsionfree modules is closed under extensions.

Note, for a cohereditary torsion theory, that a right R-module M has a codivisible cover which is torsion if and only if M is torsion.

4. **Examples.** Let $\mathscr{C}$ be a class of modules in Mod R closed under taking submodules, factor modules, extensions, direct products and direct sums. Then $\mathscr{C}$ is a torsion class for the torsion theory $(\mathscr{C}, r(\mathscr{C}))$ and a torsionfree class for the torsion theory $(l(\mathscr{C}), \mathscr{C})$. Such a class of modules is called a TTF (torsion-torsionfree) class.

We now give two examples of cohereditary torsion theories. A more detailed discussion of the results we have used concerning TTF classes can be found in [**4**].

(i) Let R be a ring with idempotent ideal I, then $\mathscr{C} = \{M \mid MI = 0\}$ is a TTF class. Thus if R is right perfect, then $(l(\mathscr{C}), \mathscr{C})$ is a cohereditary torsion theory.

(ii) If R is a left perfect ring, then $\mathscr{C} = \{M \mid [M, E(R)] = 0\}$, $E(R)$ the injective hull of R, is a TTF class. Hence if R is also right perfect, then $(l(\mathscr{C}), \mathscr{C})$ is a cohereditary torsion theory.

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Department of Mathematics, Eastern Kentucky University, Richmond, Kentucky 40475