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FAITHFUL NOETHERIAN MODULES

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ABSTRACT. The Eakin-Nagata theorem says that if T is a commutative Noetherian ring which is finitely generated as a module over a subring R, then R is also Noetherian. This paper proves a generalization of this result. However, the main interest is that the proof is very elementary and uses little more than the definition of "Noetherian".

All rings are associative and have a unit, subrings have the same unit, and modules are unitary.

A theorem due independently to Eakin [2] and Nagata [7] says that if $T=Ra_1+\cdots+Ra_k$ is a commutative ring finitely generated as a module over a subring R, then R is Noetherian if T is Noetherian. A later proof was given by Mollier [6] and there have been noncommutative generalizations by Eisenbud [3], Björk [1] and Jategaonkar and Formanek [4].

The object of this paper is to present a simple and elementary proof of the Eakin–Nagata theorem which generalizes the original version in a new direction. The proof is essentially a contraction of Eakin's proof as presented by Kaplansky in [5, Exercises 14–15, p. 54], based on the observation that much of that proof disappears if one is not "handicapped" by the hypothesis that T is a ring. More precisely, T is viewed as an R-module and the Eakin–Nagata theorem is viewed as a generalization of the basic result that a commutative ring which has a faithful Noetherian module is itself Noetherian [5, Exercise 10, p. 53].

THEOREM. *Let R be a commutative ring and $T=Ra_1+\cdots+Ra_k$ a faithful finitely generated left R-module which satisfies the ascending chain condition on "extended submodules" AT, where A is an ideal in R. Then T is a Noetherian R-module and hence R is a Noetherian ring.*

PROOF. Suppose conversely that T is not a Noetherian R-module.

Step I. Let AT be an extended submodule of T maximal with respect to the property: T/AT is not a Noetherian R-module. Then $R/\mathrm{Ann}(T/AT)$ and T/AT satisfy the hypothesis of the theorem, but T/AT is not

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Noetherian. Hence we may replace R by $R/\mathrm{Ann}(T/AT)$ and T by T/AT and thus assume that T/BT is Noetherian whenever BT is a nonzero extended submodule of T.

Step II. Suppose U is any submodule of T. Then T/U is a faithful R-module iff for each nonzero $r \in R$ there is at least one of $a_1, \cdots, a_k$ (depending on r) such that $ra_i \notin U$. This latter property is inductive, so T has a maximal submodule U with respect to the property: T/U is a faithful R-module.

Now R and T/U satisfy the hypotheses of the theorem. If T/U were Noetherian then R would be Noetherian, since T/U is a faithful R-module, and then T would be Noetherian, a contradiction. Hence T/U is not Noetherian, and we may replace T by T/U.

Step III. Summarizing Steps I and II, we may assume that

(1) T is not a Noetherian R-module.

(2) If AT is a nonzero extended submodule of T, T/AT is a Noetherian R-module.

(3) If U is a nonzero submodule of T, T/U is not a faithful R-module.

Now suppose U is any nonzero submodule of T. T/U is not a faithful R-module so there is a nonzero $r \in R$ such that $rT \subseteq U$. rT is an extended submodule so T/rT is Noetherian and thus T/U is Noetherian. Hence T is Noetherian, since every proper quotient of T is Noetherian and this contradiction completes the proof.

If T is a ring (not necessarily commutative) and A is an ideal of R, then AT is the right ideal generated by A, and if R is central in T, then AT is the two-sided ideal of T generated by A. Thus the theorem yields noncommutative generalizations of the original Eakin–Nagata theorem. These are stated below and are due to Björk, who proved the above theorem with the additional hypothesis that T is a ring.

COROLLARY (BJÖRK [1]). *Suppose $T = Ra_1 + \cdots + Ra_k$ is a ring, where R is a commutative subring of T.*

(1) *If T satisfies* ACC *on extended right ideals, then R is Noetherian.*

(2) *If R is central in T and T satisfies* ACC *on extended two-sided ideals, then R is Noetherian.*

Björk has asked whether R is Noetherian if T is *left* Noetherian. This is proved in [4] using the theory of polynomial identity rings, and the theorem of this paper also plays a role.

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