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PRODUCT INTEGRAL APPROXIMATIONS OF SOLUTIONS TO LINEAR OPERATOR EQUATIONS[1]

ROBERT H. MARTIN, JR.

ABSTRACT. In the paper we develop a class of iterative methods for approximating solutions of linear operator equations in a Banach space. The main techniques involve a product integral representation of solutions to linear Stieltjes integral equations, a variation of parameters formula, and the asymptotic convergence of solutions to the homogeneous integral equation.

Let E be a real or complex Banach space and let $|\cdot|$ denote the norm on E. Denote by $\mathscr{L}(E)$ the Banach algebra of all bounded linear transformations from E into E with the norm $\|\cdot\|$ on $\mathscr{L}(E)$ defined by $\|A\|=\sup\{|Ax|:|x|\leqq 1\}$. In [1] F. E. Browder and W. V. Petryshyn show that if T is a member of $\mathscr{L}(E)$ which is asymptotically convergent (i.e., $\lim_{n\to\infty} T^n x$ exists for each x in E), I is the identity of $\mathscr{L}(E)$, and z is in the range of $I-T$, then the iterative process

$$x_{n+1} = Tx_n + z \qquad (n = 0, 1, 2, \cdots) \tag{1}$$

converges to a solution x^* of the equation

$$x - Tx = z, \tag{2}$$

for any initial approximation x_0 in E. In [3] and [4] W. G. Dotson extends this result to a more general class of iterative processes by using mean ergodic theorems (see also D. G. DeFigueiredo and L. A. Karlovitz [2]). It is the purpose of this note to extend the results of [1] by using the theory of product integrals and Stieltjes integral equations developed by J. S. Mac Nerney in [6] and [7].

Throughout this paper we assume that S is a subset of $[0, \infty)$ such that $0 \in S$ and $\sup\{t:t \in S\}=\infty$. Also, g is an increasing function from S into $[0, \infty)$ such that $g(0)=0$ and $\sup\{g(t):t \in S\}=\infty$. For notational convenience, let $\Delta=\{(s, t) \in S\times S:s\leqq t\}$. If $(s, t) \in \Delta$ and $u=(u_k)_0^n$ is a

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finite sequence in S, then u is said to be a subdivision of (s, t) if $u_0=s$, $u_n=t$, and $u_{k-1}\leqq u_k$ for $k=1, \cdots, n$. If f is a function from S into E and (s, t) is in Δ, then

$$(R)\int_s^t f(\cdot)\,dg \quad \text{and} \quad (L)\int_s^t f(\cdot)\,dg$$

denote the limits, in the sense of refinements of subdivisions, of members of E of the form

$$\sum_1^n f(u_k)[g(u_k)-g(u_{k-1})] \quad \text{and} \quad \sum_1^n f(u_{k-1})[g(u_k)-g(u_{k-1})]$$

respectively, where $(u_k)_0^n$ is a subdivision of (s, t). If $\{F_k : k=1, \cdots, n\}$ is a family of functions from E into E and x is in E, then $\prod_1^n [F_k]x$ denotes the left to right (functional composition) product $F_1 \cdot F_2 \cdots F_n x$. If F is a function from E into E, x is in E, and $(s, t) \in \Delta$, then ${}_s\prod^t [I+dgF]x$ denotes the limit, in the sense of refinements of subdivisions, of members of E of the form $\prod_1^n [I+(g(u_k)-g(u_{k-1}))F]x$ where $(u_k)_0^n$ is a subdivision of (s, t).

Now let A be a member of $\mathscr{L}(E)$ and for each $(s, t) \in \Delta$ define the member $W(s, t)$ of $\mathscr{L}(E)$ by

$$W(s, t)x = {}_s\prod^t [I + dgA]x \tag{3}$$

for each $x \in E$. It is shown in [**6**, Theorems 3.3 and 3.4] that W is well defined, and for each x in E and (s, t) in Δ,

$$W(s, t)x = x + (R)\int_s^t AW(\cdot, t)x\,dg \tag{4}$$

and

$$W(s, t)x = x + (L)\int_s^t W(s, \cdot)Ax\,dg. \tag{4$'$}$$

Since $A[I+(g(u)-g(v))A]=[I+(g(u)-g(v))A]A$ for all $(u, v) \in \Delta$, it follows easily that $AW(s, t)=W(s, t)A$ for all $(s, t) \in \Delta$. Also, let z be in E and let $A+z$ denote the (affine) mapping F from E into E defined by $Fx=Ax+z$. For each $(s, t) \in \Delta$ and $x \in E$ define

$$M(s, t)x = {}_s\prod^t [I + dg(A + z)]x. \tag{5}$$

It follows from the results of Mac Nerney [7, Theorem 1.1 and Corollary 2.1] (see, in particular, the remark on p. 637 of [7]) that M is well defined and

$$M(s, t)x = x + (R)\int_s^t AM(\cdot, t)x\,dg + (g(t) - g(s))z \tag{6}$$

for all $(s, t) \in \Delta$ and $x \in E$.

The first lemma is crucial for our analysis and is essentially a variation of parameters formula for the solutions of (6). For further results of this type see D. L. Lovelady [**5**, Lemma 7] and J. A. Reneke [**8**].

LEMMA 1. *Suppose that W is defined by* (3) *and M is defined by* (5). *Then*

$$M(s,t)x = W(s,t)x + (R)\int_s^t W(\cdot\,,t)z\,dg \tag{7}$$

for all (s,t) in Δ and x in E.

INDICATION OF PROOF. For each $(s,t)\in\Delta$ and $x\in E$ let $M'(s,t)x$ be defined by the right side of (7). It is easy to see that $M'(\cdot\,,t)x$ is of bounded variation on the S interval $[0,t]$. Let

$$P = x + (R)\int_s^t AM'(\cdot\,,t)x\,dg + (g(t)-g(s))z.$$

If $(u_k)_0^n$ is a subdivision of (s,t) and $\delta_k g = g(u_k)-g(u_{k-1})$ for $k=1,\cdots,n$, then

$$\begin{aligned} P &\sim x + \sum_1^n A\left[W(u_k,t)x + (R)\int_{u_k}^t W(\cdot\,,t)z\,dg\right]\delta_k g + \sum_1^n z\delta_k g \\ &= x + \sum_1^n AW(u_k,t)x\delta_k g + \sum_1^n \left[z + A(R)\int_{u_k}^t W(\cdot\,,t)z\,dg\right]\delta_k g. \end{aligned}$$

Also, $z+A(R)\int_{u_k}^t W(\cdot\,,t)z\,dg = z+(R)\int_{u_k}^t AW(\cdot\,,t)z\,dg = W(u_k,t)z$ by (4); so

$$\begin{aligned} P &\sim x + \sum_1^n AW(u_k,t)x\delta_k g + \sum_1^n W(u_k,t)z\delta_k g \\ &\sim x + (R)\int_s^t AW(\cdot\,,t)x\,dg + (R)\int_s^t W(\cdot\,,t)z\,dg \\ &= W(s,t)x + (R)\int_s^t W(\cdot\,,t)z\,dg = M'(s,t)x. \end{aligned}$$

Using the above estimates, it easily follows that

$$M'(s,t)x = x + (R)\int_s^t AM'(\cdot\,,t)x\,dg + (g(t)-g(s))z.$$

Since the solution to (6) is unique by [**7**, Corollary 2.1], we have that $M'=M$ and the lemma is proved.

The function W from Δ into $\mathscr{L}(E)$ defined by (3) is said to be *asymptotically convergent* if $\lim_{t\to\infty} W(0,t)x$ exists for each x in E. Here and in the remainder of this paper, $\lim_{t\to\infty}$ denotes the limit as t tends to infinity through values in S. Also, if W is asymptotically convergent, let

$$Qx = \lim_{t \to \infty} W(0, t)x \tag{8}$$

for each x in E.

Lemma 2. *Suppose that W is asymptotically convergent and Q is defined by* (8). *Then*

(i) *Q is in $\mathscr{L}(E)$ and $Q^2 = Q$;*
(ii) *$AQ = QA = 0$ and $W(s, t)Q = QW(s, t) = Q$ for all (s, t) in Δ; and*
(iii) *the range of Q is the null space of A.*

Indication of Proof. Using (4)′ with $s=0$, we have that

$$\lim_{t \to \infty} (L)\int_0^t W(0, \cdot)Ax\, dg = Qx - x.$$

Thus, since $\lim_{s\to\infty} W(0, s)Ax = QAx$ and $\lim_{s\to\infty} g(s) = \infty$, it is straightforward to show that $QAx = \theta$ (where θ is the zero of E). Since $W(0, t)A = AW(0, t)$ and $W(s, t)x = x$ if $Ax = \theta$, part (ii) is immediate. Also, Q is in $\mathscr{L}(E)$ by the uniform boundedness theorem and $Q^2 = \lim_{t\to\infty} W(0, t)Q = Q$ by (ii). Hence (i) is true. Finally, if x is in E and $Ax = \theta$, then $W(0, t)x = x$ for all t in S and we have that $Qx = x$. Part (iii) now follows easily and the proof is complete.

Letting $A = T - I$, we have that equation (2) becomes

$$Ax + z = \theta, \tag{9}$$

where θ is the zero of E. With the approach used in this paper, it is more convenient to study the existence of solutions to (9) as opposed to (2).

Theorem 1. *Suppose that W is defined by* (3), *M is defined by* (5), *and W is asymptotically convergent. If z is in the range of A and x_0 is in E, then $x^* = \lim_{t\to\infty} M(0, t)x_0$ exists and $Ax^* + z = \theta$.*

Proof. Let $y \in E$ be such that $Ay = z$. Using (7), (4), and the fact that $W(s, t)A = AW(s, t)$ we have that

$$\begin{aligned} M(0, t)x_0 &= W(0, t)x_0 + (R)\int_0^t W(\cdot, t)Ay\, dg \\ &= W(0, t)x_0 + (R)\int_0^t AW(\cdot, t)y\, dg \\ &= W(0, t)x_0 + W(0, t)y - y. \end{aligned}$$

Hence $x^* = \lim_{t\to\infty} M(0, t)x_0 = Qx_0 + Qy - y$ where Q is defined by (8). Thus $Ax^* + z = A(-y) + z = \theta$ by (ii) of Lemma 2.

If $(x_n)_1^\infty$ is a sequence in E which converges weakly to a member x of E, we write $w\text{-}\lim_{n\to\infty} x_n = x$. Also, for the proof of our next theorem, we use the fact that if A is in $\mathscr{L}(E)$ and $w\text{-}\lim_{n\to\infty} x_n = x$, then $w\text{-}\lim_{n\to\infty} Ax_n = Ax$.

THEOREM 2. *Suppose that W is defined by* (3), *M is defined by* (5), *and W is asymptotically convergent. In addition, suppose that there is an x_0 in E and a sequence $(t_n)_1^\infty$ in S such that* $\lim_{n\to\infty} t_n = \infty$ *and* $w\text{-}\lim_{n\to\infty} M(0, t_n)x_0$ *exists. Then z is in the range of A and the conclusions of Theorem* 1 *are valid.*

PROOF. Using (7) and the fact that $QW(s, t) = Q$ (see (ii) of Lemma 2), we have that

$$\begin{aligned} QM(0, t_n)x_0 &= QW(0, t_n)x_0 + Q(R)\int_0^{t_n} W(\cdot\,, t_n)z\,dg \\ &= Qx_0 + (R)\int_0^{t_n} Qz\,dg = Qx_0 + g(t_n)Qz. \end{aligned}$$

Thus $Qz = \theta$ since $\lim_{n\to\infty} g(t_n) = \infty$. Again using (7),

$$\begin{aligned} AM(0, t_n)x_0 &= AW(0, t_n)x_0 + A(R)\int_0^{t_n} W(\cdot\,, t_n)z\,dg \\ &= AW(0, t_n)x_0 + (R)\int_0^{t_n} AW(\cdot\,, t_n)z\,dg \\ &= AW(0, t_n)x_0 + W(0, t_n)z - z, \end{aligned}$$

where (4) was employed to obtain the last identity. Hence, if $x^* = w\text{-}\lim_{n\to\infty} M(0, t_n)x_0$,

$$A(-x^*) = -w\text{-}\lim_{n\to\infty} AM(0, t_n)x_0 = -AQx_0 - Qz + z = z,$$

and the assertions of Theorem 2 follow.

As an example of the above results, let S be the set of nonnegative integers, let $(\lambda_k)_1^\infty$ be a sequence of positive numbers such that $\sum_1^\infty \lambda_k = \infty$, and let $g(n) = \sum_1^n \lambda_k$ for each $n \in S$ (where $\sum_1^0 \lambda_k = 0$). If we let $A = T - I$, then it is easy to see that

$$W(0, n)x = \prod_1^n [(1 - \lambda_k)I + \lambda_k T]x,$$

and

$$M(0, n)x = \prod_1^n [(1 - \lambda_k)I + \lambda_k(T + z)]x$$

for each $n \in S$ and $x \in E$. In particular, W and M satisfy the recursion formulas

$$W(0, n+1)x = (1-\lambda_{n+1})W(0, n)x + \lambda_{n+1}TW(0, n)x,$$

and

$$M(0, n+1)x = (1-\lambda_{n+1})M(0, n)x + \lambda_{n+1}TM(0, n)x + \lambda_{n+1}z$$

for each $n \in S$. If there is a number λ in $(0, 1]$ such that $\lambda_k = \lambda$ for all $k \geqq 1$, we have that

$$W(0, n)x = [(1-\lambda)I + \lambda T]^n x,$$

and

$$M(0, n)x = [(1-\lambda)I + \lambda(T+z)]^n x$$

for each $n \in S$ and $x \in E$. The variation of parameter formula (7) shows also that $M(0, n+1)x = [(1-\lambda)I + \lambda T]^{n+1}x + \sum_0^n [(1-\lambda)I+\lambda T]^k z$. In particular, setting $\lambda = 1$, we have the result of Browder and Petryshyn [**1**]. If $S = [0, \infty)$ and $g(t) = t$ for all $t \in S$, then $W(s, t) = \exp((t-s)A)$ for all $(s, t) \in \Delta$ and $M(s, \cdot)x$ is the solution u to the differential equation $u'(t) = Au(t) + z$ which satisfies $u(s) = x$. In this case (7) is the classical variation of parameters formula.

In closing, let us point out that a theory involving the mean asymptotic convergence of W can also be developed using Stieltjes integral equations. Define the function P from S into $\mathscr{L}(E)$ by $P(0) = I$ and, for each $t \in S$, $t \neq 0$, let

$$P(t)x = g(t)^{-1}(L)\int_0^t W(0, \cdot)x\, dg$$

for all $x \in E$. Then W is said to be *mean asymptotically convergent* if $Qx = \lim_{t\to\infty} P(t)x$ exists for each $x \in E$. Using techniques similar to the above, one can establish theorems on mean convergence analogous to Theorems 1 and 2. However, this type of theory is subsumed directly by the results of Dotson [**3**], [**4**].

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DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NORTH CAROLINA 27607

Current address: Mathematics Division, University of Sussex, Falmer, Brighton, England