PRIME IDEALS IN UNIFORM ALGEBRAS

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Abstract. A uniform algebra on a compact metric space has infinite Krull dimension and exactly $2^\omega$ nonmaximal prime ideals.

A subalgebra of the continuous, complex-valued functions $C(X)$ on a compact Hausdorff space $X$ which separates the points of $X$, contains the constant functions and is closed in the uniform norm $||\cdot||_\infty$ is called a uniform algebra. This topic has been elaborated for more than two decades (e.g. [4]), but with little attention to algebraic questions. In the classical disc algebra, for example, 0 is a prime ideal and no nonzero prime ideal can lie properly in any maximal ideal determined by an interior point of the disc: does the disc algebra even have a nonzero, nonmaximal prime?

An interpolation method used here implies that each boundary maximal ideal will contain $2^\omega$ such ideals, arranged in $2^\omega$ nonoverlapping infinite chains [Theorem 1]. Even more, every uniform algebra on a first countable space has at least one maximal ideal with this property [Theorem 2]; in particular, its Krull dimension is infinite.

For a subset $B$ of a uniform algebra $A$, let $Z(B)$ stand for the set of common zeros of the functions in $B$ and for $p \in X$, let $I_p$ denote the maximal ideal of functions in $A$ which vanish at $p$. Denote the Choquet boundary of $A$ by $\partial A$ [1, p. 81].

Theorem 1. Suppose $A$ is a uniform algebra and $J$ is an ideal of $A$. If $p$ is a peak point for $A$ which is not isolated in $Z(J) \cap \partial A$, there are $2^\omega$ pairwise disjoint, infinite ascending chains of prime ideals of $A$ with each prime containing $J$ and densely contained in $I_p$. In particular, krull dim $A/J = \infty$.

Proof. Choose $f \in A$ with $f(p) = 1$ and $|f(x)| < 1$ if $x \neq p$. Inductively select $p_n \in Z(J) \cap \partial A$ so that $|1 - f(p_n)| < \min\{1/n, |1 - f(p_{n-1})|\}$. $p$ is the only possible accumulation point $q$ of the set $\{p_n\}$. For $f(q)$ is an accumulation point of the distinct points $\{f(p_n)\}$; since $f(p_n) \to 1$, $f(q) = 1$ so actually $q = p$. Thus $K = \{p_n\} \cup \{p\}$ is compact, $f$ is a homeomorphism of $K$ onto $f(K)$ and $p_n \to p$.

Received by the editors December 26, 1972 and, in revised form, April 27, 1973.


Key words and phrases. Prime ideal, uniform algebra, peak point, Choquet boundary, interpolating sequence.

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$A|K$ is dense in $C(K)$. Take a Borel measure $\mu$ on $K$ which annihilates $A|K$; since $f^n \to \chi_{[n]}$ boundedly, $0 = \int_K f^n \, d\mu \to \mu([p])$. Also because $p_n \to p$, each $p_n$ has a neighborhood $V_n$ which misses $K\setminus\{p_n\}$; since $p_n$ is a strong boundary point of $A \{1, 2, 3, 4\}$, there is some $k_n \in A$ with $\|k_n\|_\infty = 1 = k_n(p_n)$ and $|k_n| < 1$ off $V_n$. Given any $\varepsilon > 0$ we can take a high enough power of $k_n$ to obtain some $g_n \in A$ with $g_n(p_n) = 1$ and $|g_n| < \varepsilon$ on $K \setminus \{p_n\}$.

$$0 = \left| \int_K g_n \, d\mu \right| \geq |\mu(\{p_n\})| - \varepsilon \|\mu\|,$$

and letting $\varepsilon \to 0$ we see that $\mu(\{p_n\}) = 0$. Thus

$$|\mu| (K) = |\mu(\{p\})| + \sum_{n=1}^\infty |\mu(\{p_n\})| = 0;$$

by the Hahn-Banach theorem, $A|K$ is dense in $C(K)$.

$K$ is a closed set which is a countable union of peak points in the weak sense, so that Glicksberg’s peak set theorem $[4, II. 12.7, p. 58]$ implies $K$ is an intersection of peak sets. Thus $A|K$ is closed in $C(K)$; in fact then, $A|K = C(K)$.

$K$ is homeomorphic to $N_\infty$, the one point compactification of the natural numbers, and composing the induced isomorphism $C(K) \cong C(N_\infty)$ with restriction $A \to C(K)$, we obtain an algebra homomorphism $\Phi$ of $A$ onto $C(N_\infty)$ such that $J \subseteq \ker \Phi$ and $\Phi(I_p) = M_\infty = \{ f \in C(N_\infty) : f(\infty) = 0 \}$. According to $[5, 14G, p. 213]$ there are $2^c$ maximal chains of prime ideals of $C_c(N_\infty)$ [the real-valued continuous functions on $N_\infty$] contained in $M'_\infty = M_\infty \cap C_c(N_\infty)$, and any two chains have only $M'_\infty$ in common. For any such chain $\mathcal{C}$, set $\mathcal{C}^* = \{ \Phi^{-1}(P+iP) : P \in \mathcal{C}, P \neq M'_\infty \}$. Since $P \to P+iP$ is a lattice preserving one-to-one correspondence between the primes of $C_c(N_\infty)$ and those of $C(N_\infty)$ $[3, 1.1]$, $\mathcal{C}^*$ is a chain of prime ideals of $A$ contained in $\Phi^{-1}(M'_\infty) = I_p$ and containing $J$; plainly if $\mathcal{D}^*$ is any other such chain, $\mathcal{D}^* \cap \mathcal{C}^* = \emptyset$. Each chain $\mathcal{C}^*$ is infinite ascending since $P \to \Phi^{-1}(P+iP)$ is a lattice preserving bijection and $\mathcal{C}$ is infinite ascending. For otherwise there is some largest $P \in \mathcal{C}$ properly contained in $M'_\infty$, and because $\mathcal{C}$ is maximal, there is no prime ideal of $C_r(N_\infty)$ strictly between $P$ and $M'_\infty$; a violation of $[2, 3.2, p. 71]$.

Finally each $Q \in \mathcal{C}^*$ is dense in $I_p$. Indeed the prime $P = \Phi(Q)$ is dense in $M_p$ $[2, 1.5, 1.8]$; given $f \in I_p$, there are $g_n \in P$ with $\|g_n - \Phi(f)\|_\infty \to 0$. Since $K$ is a peak interpolation set, there are $h_n \in A$ with $\|h_n\|_X = \|g_n - \Phi(f)\|_\infty$ and $\Phi(h_n) = g_n - \Phi(f)$. Thus $h_n + f \in \Phi^{-1}(P) = Q$ and $\|h_n + f - f\|_X \to 0$.

Of course none of the nonmaximal primes constructed above is closed. This is to be expected since in the disc algebra, for example, Rudin’s
characterization of the closed ideals [6, p. 85] implies that 0 is the only nonmaximal closed prime.

In practice a uniform algebra may only have peak points isolated in its Šilov boundary, or because no point has a countable neighborhood base, even none at all. Nevertheless we have

**Theorem 2.** A uniform algebra \( A \) on an infinite first countable space \( X \) has a maximal ideal which contains \( 2^\omega \) pairwise disjoint infinite chains of prime ideals.

**Proof.** Suppose \( \partial A \) is a discrete subspace of the Šilov boundary \( \Gamma \) of \( A \). Then each \( p \in \partial A \) is a peak point [1, 2.3.1] which is open in \( \Gamma \): if \( f \) peaks at \( p \), \( f^n \to \chi_{(p)} \) uniformly on \( \Gamma \) so \( 1-\chi_{(p)} \in \mathcal{A} \) peaks (in \( \Gamma \)) on \( \Gamma - \{p\} \). Since \( \Gamma \) is infinite [otherwise \( \mathcal{A} \subseteq \mathcal{A}[\Gamma] \) is finite-dimensional and because \( \mathcal{A} \) separates point, \( X \) is finite], \( \mathcal{F} = \{ \Gamma - F : F \subseteq \partial A \) finite\} \) is a family of nonvoid closed subsets of \( \Gamma \) with the finite intersection property, so \( P = \bigcap \mathcal{F} \) is a generalized peak set for \( \mathcal{A}[\Gamma] \). By a theorem of Bishop [1, 2.4.6, p. 105] \( P \) contains a generalized peak point \( p \in \mathcal{A}[\Gamma] = \partial A \). Thus \( p \in \Gamma - \{p\} \), a contradiction.

We conclude that \( \partial A \) contains a point \( p \) nonisolated in \( \partial A \). \( p \) is a peak point which is the limit of an infinite sequence on \( \partial A \); the result follows from the proof of Theorem 1 with \( J = 0 \).

Notice if \( X \) is actually metric (and hence separable), \( A \) will have cardinality \( c \), so that Theorem 2 implies

**Corollary 1.** A uniform algebra on an infinite metric space has infinite Krull dimension and exactly \( 2^\omega \) nonmaximal prime ideals.

The following answers a question of M. Weiss [8, p. 94].

**Corollary 2.** In a uniform algebra on an infinite first countable space, not every finitely generated ideal is principal.

**Proof.** Otherwise, the primes contained in a fixed maximal ideal form a chain [5, 14L, p. 214] in violation of Theorem 2.

**Example.** \( H^\omega \), the bounded analytic functions on the open unit disc \( \Delta \), considered as a uniform algebra on its maximal ideal space \( M \) has no peak points and no point of \( M - \Delta \) has a countable neighborhood base: Theorems 1 and 2 do not apply. Nonetheless, suppose \( q \in M - \Delta \) lies in the closure of a Carleson-Newman interpolating sequence \( S \subseteq \Delta : H^\omega | S = l^\omega = C^*(S) \). Then cl \( S \) is homeomorphic to \( \beta S \), the Stone-Čech compactification of \( S \) [6, p. 205], so that actually \( H^\omega | \beta S = C(\beta S) \). Since \( S \) is realcompact, there is some \( f \in C_r(\beta S) \) with \( f(q) = 0 \) and \( |f| > 0 \) on \( S \) [5, p. 119]. \( f \) cannot vanish on any neighborhood of \( q \) and assuming the continuum hypothesis, it follows that the maximal ideal \( M_q \) of \( C_r(\beta X) \) determined by \( q \) contains a
chain of at least \(2^c\) prime ideals \([5, 14.19, \text{p. 204}]\). Because restriction is a homomorphism of \(H^\infty\) onto \(C(\beta S)\) which takes \(I_q\) onto \(M_q + iM_q\), \(I_q\) contains a chain of \(2^c\) prime ideals of \(H^\infty\); in particular, krull dim \(H^\infty = \infty\).

Although interpolating sequences exist in profusion \([6, \text{p. 204}]\), not every \(q \in M - \Delta\) lies in the closure of such a set; in fact Hoffman has shown this happens exactly when the Gleason part for \(q\) is nontrivial \([7, 5.5, \text{p. 101}]\). Since \(H^\infty\) is logmodular on the maximal ideal space \(X\) of \(L^\infty\), each point of \(X\) is a one point part; and there are others \([4, \text{Example 3, p. 162}]\). The prime structure at these points is not known, but things are clear elsewhere: if \(q \in \Delta\), a routine order of zero argument shows that \(0\) is the only nonmaximal prime of \(H^\infty\) in \(I_q = (z - q)H^\infty\); if \(q \in M - \Delta\) has a nontrivial part, \(I_q\) contains exactly \(2^c\) nonmaximal primes. Indeed, Carleson’s corona theorem makes \(M\) separable, so that \(H^\infty\) has cardinality \(c\): even \(H^\infty\) has exactly \(2^c\) nonmaximal primes.

For \(|\lambda| = 1\) the fiber \(M^\lambda = \{\phi \in M : \phi(z) = \lambda\}\) is a peak set for \(H^\infty\), so that \(A_\lambda = H^\infty|_{M^\lambda}\) is a uniform algebra. There is an embedding \(\psi : \Delta \to M^\lambda\) so that \(A_\lambda[\psi(\Delta)] \cong H^\infty\) \([6, \text{p. 168}]\), and the composite \(A_\lambda \to A_\lambda[\psi(\Delta)] \to H^\infty \xrightarrow{\psi} C(\beta S)\) makes \(C(\beta S)\) a homomorphic image of \(A_\lambda\). Therefore \(A_\lambda\) will also contain exactly \(2^c\) nonmaximal prime ideals and will have infinite Krull dimension.

REFERENCES


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