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CONTINUITY OF CERTAIN CONNECTED FUNCTIONS AND MULTIFUNCTIONS

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ABSTRACT. In this paper it is proved that if X is a 1st countable, locally connected, T_1-space and Y is a σ-coherent, sequentially compact T_1-space, then any nonmingled connectedness preserving multifunction from X onto Y with closed point values and connected inverse point values is upper semicontinuous. It follows that any monotone, connected, single-valued function from X onto Y is continuous. Let X be as above and let Y be a sequentially compact T_1-space with the property that if a descending sequence of connected sets has a nondegenerate intersection, then this intersection must contain at least three points. If f is a monotone connected single-valued function from X onto Y, then f is continuous. An example of a noncontinuous monotone connected function from a locally connected metric continuum onto an hereditarily locally connected metric continuum is given.

In [1] and [2] conditions are given under which an open monotone connected function is continuous. This paper is concerned with conditions under which a monotone connected function is continuous. As Example 2 below shows, a monotone connected function from an hereditarily locally connected metric continuum onto a nonlocally connected metric continuum is not necessarily continuous, and Example 3 shows that a monotone connected function from a locally connected metric continuum onto an hereditarily locally connected metric continuum is not necessarily continuous. It is an open question as to whether or not such a function is continuous if both the domain and range are hereditarily locally connected metric continua.

Some definitions will now be recalled. A multifunction $F: X \to Y$ is upper semicontinuous at a point $p \in X$ if for any open set $V \subset Y$, with $F(p) \subset V$, there is an open set $U \subset X$, with $p \in U$, such that $F(U) \subset V$, and F is nonmingled provided for any $p, q \in X$, either $F(p) = F(q)$ or $F(p) \cap F(q) = \varnothing$.

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See [6] for other terminology and properties of multifunctions. A single-valued function is monotone if point inverses are connected and is connected if the function preserves connectedness. This same terminology will be used for multifunctions also. A space Y is σ-coherent provided any descending sequence of connected sets has a connected intersection. Finally, $\limsup A_n$ will denote the upper limit of a sequence $\{A_n\}$ of sets as defined on p. 337 of [3].

THEOREM 1. *Let X be a 1st countable, locally connected T_1-space and Y a σ-coherent, sequentially compact T_1-space. If F is a nonmingled connected multifunction from X onto Y with closed point values, such that F^{-1} has connected point values, then F is upper semicontinuous.*

PROOF. Suppose F is not upper semicontinuous at the point $p \in X$. Then there is an open set $V \subset Y$ such that $F(p) \subset V$, but for any open set $U \subset X$, with $p \in U$, there is some $x \in U$ such that $F(x)$ is not contained in V. Let $\{U_n\}$ be a countable base at p consisting of open connected sets with $U_{n+1} \subset U_n$ for all n. For each n, let $p_n \in U_n$ such that $F(p_n)$ is not contained in V, and let $A_n = (Y-V) \cap F(p_n)$. Then $\{A_n\}$ is a sequence of sets all lying in the closed set $Y-V$. Since Y is sequentially compact, there is a point $q \in (Y-V) \cap \limsup A_n$. If $q \in F(p_n)$ for all but finitely many n, then it can be assumed that $q \in F(p_n)$ for all n. Thus, $F(p_1) = F(p_n)$ for all n, since F is nonmingled. Therefore, $F(p_1) \subset F(U_n)$ for all n, which implies that $q \in K = \bigcap_{n=1}^{\infty} F(U_n)$. Also, $F(p) \subset V$ and $q \in (Y-V)$. Hence, $F(p)$ and q are separated in K since $F(p) \subset K$. But K is connected since Y is σ-coherent. Therefore, there is a point y in $K-(F(p) \cup q)$. Since $y \in F(U_n)$ for all n, $F^{-1}(y) \cap U_n \neq \varnothing$ for all n. Therefore p is a limit point of $F^{-1}(y)$. But by Corollary D_2 of [6], $F^{-1}(y)$ is closed. Hence, $p \in F^{-1}(y)$. This implies $y \in F(p)$, which is a contradiction. Thus, it must be the case that $q \notin F(p_n)$ for infinitely many n. Now $F(p_j) \subset F(U_n)$ for all $j \geqq n$, and every neighborhood of q intersects $F(p_j)$ for infinitely many j. Thus, q is a limit point of $F(U_n)$ for every n. Therefore, $\{F(U_n) \cup q\}$ is a descending sequence of connected sets, and by hypothesis $K = \bigcap_{n=1}^{\infty} (F(U_n) \cup q)$ is connected. But again, $F(p)$ and q are separated in K and therefore there is a point y in $K-(F(p) \cup q)$. This leads to the same contradiction as before. Hence, F must be upper semicontinuous.

COROLLARY. *If X and Y are as in Theorem 1 and f is a monotone, connected, single-valued function from X onto Y, then f is continuous.*

PROOF. A single-valued function is nonmingled, and since Y is T_1, f has closed point values. Also, f monotone means f^{-1} has connected point values.

THEOREM 2. *Let X be as in Theorem 1 and let Y be a sequentially compact T_1-space with the property that if a descending sequence of connected sets has a nondegenerate intersection, then this intersection must contain at least three points. If f is a monotone connected single-valued function from X onto Y, then f is continuous.*

PROOF. The proof proceeds exactly as in the proof of Theorem 1. The set K has at least two points $f(p)$ and q, so by hypothesis must contain a third point y distinct from $f(p)$ and q. This leads to the same contradiction as in the proof of Theorem 1. Thus, f must be upper semicontinuous. But this is equivalent to continuity since f is single-valued.

The following example shows that not every hereditarily locally connected metric continuum has the property given in the hypothesis of Theorem 2, that if a descending sequence of connected sets has a nondegenerate intersection, then the intersection has at least three points.

EXAMPLE 1. This is a modification of an example given in [4, p. 284]. In the plane let C_{nk} denote the semicircle given by

$$(x-(2k-1)/2^n)^2+y^2=1/4^n, \qquad y \geqq 0.$$

Denote by L_{nk} the straight line segment given by $x=(2k-1)/2^n$, $0 \leqq y \leqq 1/2^n$. Let Q_{nk} denote the semicircle given by

$$(x-(2k-1)/(2\cdot 3^n))^2+y^2=1/(4\cdot 9^n), \qquad y \leqq 0.$$

Denote by R_{nk} the straight line segment given by

$$x=(2k-1)/(2\cdot 3^n), \qquad -(2k-1)/(2\cdot 3^n) \leqq y \leqq 0.$$

Let H_n denote the union of all the C_{nk} and L_{nk}, $1 \leqq k \leqq 2^{n-1}$, and denote by K_n the union of all the Q_{nk} and R_{nk}, $1 \leqq k \leqq 3^n$. Finally, let X denote the union of all the H_n and all the K_n, n varying over all positive integers, along with the interval $I=[0, 1]$. Then X is an hereditarily locally connected continuum. Let D denote the set of end points of all upper semicircles C_{nk} and T the set of end points of all lower semicircles Q_{nk}, and let $p=(0, 0)$ and $q=(1, 0)$. Let $U_1=(X-I)\cup D\cup T$ and for $n \geqq 1$, let $U_{n+1}=(U_n-(H_n\cup K_n))\cup\{p, q\}$. Then $\{U_n\}$ is a descending sequence of connected sets whose intersection consists of just the points p and q.

In the following example, an hereditarily locally connected σ-coherent continuum is mapped by a noncontinuous one-to-one connected function f onto a nonlocally connected, 1st countable continuum. The function f^{-1} is also connected and noncontinuous. Since all the hypotheses of Theorem 1 is satisfied for f^{-1} except local connectedness of the domain of f^{-1}, this property is necessary in Theorem 1.

EXAMPLE 2. Choose a polar coordinate system on the plane and for each positive integer n, let L_n denote the segment $\{(r, (\pi/2)/n) | 0 \leqq r \leqq 1/n\}$. Let $X = \bigcup_{n=1}^{\infty} L_n$. Now choose a rectangular coordinate system on the plane with the same origin, and for each positive integer n, let S_n denote the segment $\{(x, y) | 0 \leqq x \leqq 1, y = x/n\}$, and S_0 the segment $\{(x, 0) | 0 \leqq x \leqq 1\}$. Then $Y = \bigcup_{n=0}^{\infty} S_n$ is a nonlocally connected non-σ-coherent metric continuum. Define a function f from X onto Y as follows: For each $n \geqq 1$, let $p_n = (1/n, (\pi/2)/n)$ in polar coordinates, and $q_n = (1, 1/n)$ in rectangular coordinates. Let $p_0 = (0, 0)$ and $q_0 = (1, 0)$. Let f be the function that takes L_n linearly onto S_{n-1} such that $f(p_0) = p_0$ and $f(p_n) = q_{n-1}$, $n \geqq 1$. Then f is one-to-one, connected, and not continuous at p_0. Also, the function f^{-1} taking Y onto X satisfies all the hypothesis of Theorem 1 except that the domain Y is not locally connected, and f^{-1} is not continuous at q_0. Thus, local connectedness of the domain space is necessary in Theorem 1.

Finally, the following is an example of a noncontinuous, monotone, connected function from a locally connected metric continuum onto an hereditarily locally connected metric continuum.

EXAMPLE 3. Choose a rectangular coordinate system on Euclidean 3-space E_3 and let $p = (0, 0, 0)$ and $q = (1, 0, 0)$. Also, choose a spherical coordinate system, as defined on p. 355 of [5], with origin p. Thus $q = (1, 0, \pi/2)$ in spherical coordinates. For each n, let

$$p_n = (1/n, \pi/2, \pi/(2n)) \quad \text{and} \quad q_n = ((2^n - 1)/2^n, 0, \pi/2)$$

in spherical coordinates. If a and b are end points of a straight line segment, let ab denote the ordered segment from a to b. Let P_n denote the plane determined by the three points p, p_n, q_n, L_n the ordered segment from p_n to q_n, and S_n the ordered segment from p to p_n.

Define inductively a sequence of finite ordered subsets of pq as follows: Let $H_2 = \{x_{21}, x_{22}\}$, where x_{21} and x_{22} are the mid points, respectively, of the segments pq_1 and q_1q_2. Let $H_3 = \{x_{31}, x_{32}, x_{33}, x_{34}, x_{35}\}$, where these points are, respectively, the mid points of the segments px_{21}, $x_{21}q_1$, q_1x_{22}, $x_{22}q_2$, q_2q_3. Assuming H_{n-1} has been defined, let $H_n = \{x_{n1}, \cdots, x_{nk_n}\}$, where the x_{ni} are the mid points of the ordered collection of segments of pq_n determined by the points of H_{n-1}. The order of the listing in H_n is by increasing distance from p, where x_{n1} is the mid point of the segment $px_{n-1,1}$ and x_{nk_n} is the mid point of the segment $q_{n-1}q_n$. Note that the union of the H_n's is a countable dense subset of pq.

For each $n \geqq 2$ and each j, $1 \leqq j \leqq k_n$, let L_{nj} denote the line segment parallel to L_n with one end point at x_{nj} and the other end point, denoted by s_{nj}, on the segment S_n. Thus, $\{s_{n1}, \cdots, s_{nk_n}\}$ is a finite subset of S_n and each s_{nj} is joined to the corresponding x_{nj} by the segment L_{nj} lying in the plane P_n and parallel to L_n.

Let $K_n = (\bigcup_{i=2}^{n} H_i) \cup \{q_1, \cdots, q_{n-1}\}$. For each point $y \in K_n$, let L_y denote the segment parallel to S_n with one end point at y and the other end point on L_n. Thus, each L_y lies in the plane P_n. Let $Q_1 = pq \cup pp_1 \cup p_1q_1$, and for $n \geqq 2$, let Q_n denote the union of the segments pq, S_n, L_n, L_y, $y \in K_n$, and L_{nj}, $1 \leqq j \leqq k_n$. Then $X = \bigcup_{n=1}^{\infty} Q_n$ is a locally connected continuum.

REMARK. Let R_n denote the plane given by $z = y/n$ in rectangular coordinates, where n is a positive integer. It is to be understood that any arc of a circle subsequently described with end points on pq and lying in R_n is to have altitude less that one-half of the minimum of the altitudes of all such previously described arcs and is disjoint from all such arcs except possibly at some end points.

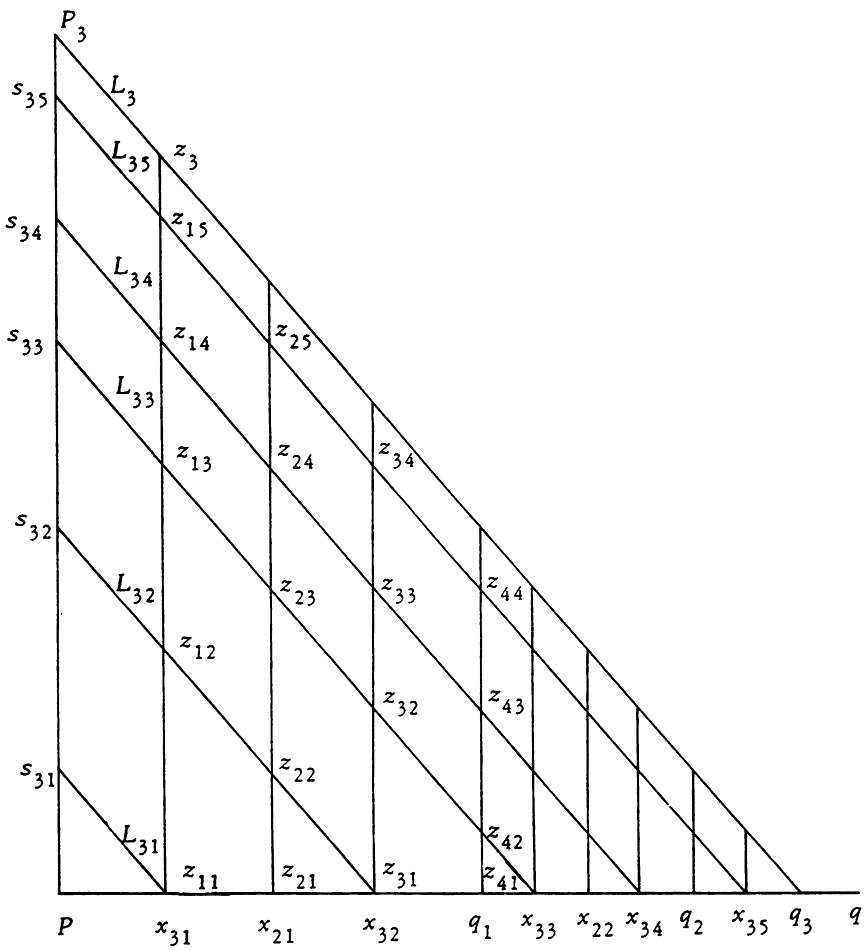

Figure 1

Define a function f on X into E_3 as follows: It will be sufficient to define f on each Q_n. Note that $Q_1 = pq \cup pp_1 \cup p_1q_1$. The function f will be the identity function on pq. Let f take pp_1 homeomorphically onto an arc of a circle of radius one-half the distance from p to q_1 lying in R_1 with end points p and q_1 with $f(p_1)=q_1$, and let $f(p_1q_1)=q_1$. The function f will now be defined on Q_3 and it will be clear that the same process can be used on any Q_n. The set K_3 consists of the points x_{31}, x_{21}, x_{32}, q_1, x_{33}, x_{22}, x_{34}, q_2, x_{35}, and Q_3 is

$$pq \cup S_3 \cup L_3 \cup \left(\bigcup_{i=1}^{5} L_{3i}\right) \cup \left(\bigcup \{L_y \mid y \in K_3\}\right).$$

Define f on S_3, which is $ps_{31} \cup s_{31}s_{32} \cup s_{32}s_{33} \cup s_{33}s_{34} \cup s_{34}s_{35} \cup s_{35}p_3$, as follows: Let f take ps_{31} homeomorphically onto an arc of a circle lying in R_3 where the arc has end points p and x_{31} and altitude one-half the distance from p to x_{31}, and $f(s_{31})=x_{31}$. Let f take $s_{31}s_{32}$ homeomorphically onto an arc of a circle lying in R_3 where the arc has end points x_{31}, x_{32} and is subject to the conditions in the above remark, and $f(s_{32})=x_{32}$. Map segments $s_{32}s_{33}$, $s_{33}s_{34}$, and $s_{34}s_{35}$ in a similar manner. Let f take $s_{35}p_3$ homeomorphically onto an arc of a circle lying in R_3 where the arc has end points x_{35}, q_3 and subject to the conditions in the above remark, and $f(s_{35})=x_{35}$ and $f(p_3)=q_3$. Let $f(L_3)=q_3$ and $f(L_{3i})=x_{3i}$, $1 \leqq i \leqq 5$.

For $y=x_{31} \in K_3$, L_y intersects L_{3j} at, say z_{1j}, $1 \leqq j \leqq 5$, where $z_{11}=x_{31}$, and L_y intersects L_3 at, say z_3. Map $x_{31}z_{12}$ homeomorphically onto an arc of a circle lying in R_3 satisfying the conditions in the above remark, with end points x_{31}, x_{32}, where $f(z_{12})=x_{32}$. Let f take $z_{12}z_{13}$ homeomorphically onto an appropriate arc of a circle lying in R_3 with end points x_{32}, x_{33}, where $f(z_{13})=x_{33}$. Map $z_{13}z_{14}$ homeomorphically onto an appropriate arc

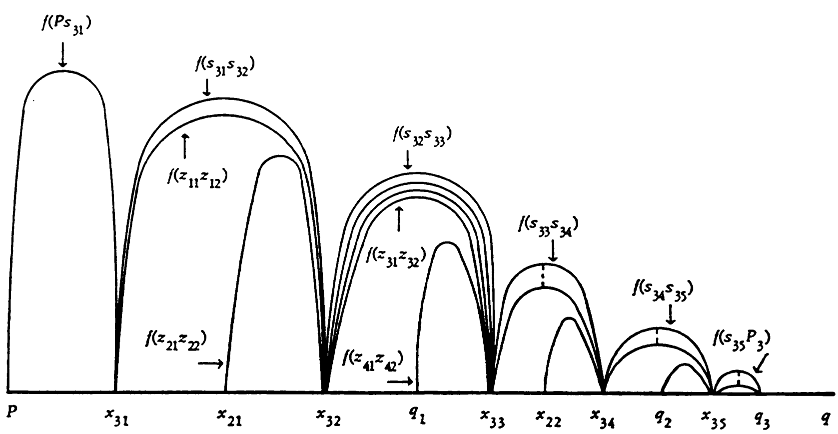

Figure 2

of a circle lying in R_3 with end points x_{33}, x_{34}, where $f(z_{14})=x_{34}$. Map $z_{14}z_{15}$ in a similar manner. Let f take $z_{15}z_3$ homeomorphically onto an appropriate arc of a circle lying in R_3 with end points x_{35}, q_3, where $f(z_{15})=x_{35}$ and $f(z_3)=q_3$. This defines f on L_y, where $y=x_{31}$. Define f similarly on each L_y, $y \in K_3$, where the arcs chosen are subject to the conditions in the above remark. The function is now defined on Q_3. The set Q_3 and its image $f(Q_3)$ are displayed in Figures 1 and 2, respectively. In Figure 2 the curves are supposed to represent circular arcs. Define f in a similar manner on all the Q_n, where the initial arcs are to be chosen in R_n having radius one-half the distance from p to x_{n1}, and subject to the conditions in the above remark.

The resulting function f is monotone since the inverse of a point is either a point or a segment, f maps connected sets onto connected sets, $f(X)$ is a hereditarily locally connected continuum in E_3, but f is not continuous at p since the sequence $\{p_n\}$ converges to p and the sequence $\{f(p_n)\}$ converges to $q \neq f(p)$.

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