COHOMOLOGY, MAXIMAL IDEALS, AND POINT EVALUATIONS[1]

ALEXANDER NAGEL[2]

ABSTRACT. We consider algebras A of continuous complex valued functions, which are given as the set of global sections of a sheaf $\mathscr{S}$ on a topological space X. Under the hypothesis that all the higher cohomology groups of the sheaf are zero, we investigate the relationship between ideals in A, kernels of algebra homomorphisms of A into the complex numbers C, and sets of functions vanishing at a point of X. As applications, we obtain some simple proofs of theorems about ideals in certain algebras of holomorphic functions.

Let X be a topological space, and let $\mathscr{S}$ be a sheaf of local C-algebras on X. We shall assume:

(a) For all $x \in X$, the maximal ideal of the stalk $\mathscr{S}_x$ is m_x, and the composition $C \to \mathscr{S}_x \to \mathscr{S}_x/m_x$ is an isomorphism.

(b) For every $f \in \Gamma(X, \mathscr{S})$, the associated complex valued function $\hat{f}$ is continuous, where $\hat{f}(x)$ is the residue class of the germ of f at x in $\mathscr{S}_x/m_x$.

(c) For all $q \geq 1$, $H^q(X, \mathscr{S}) = (0)$.

THEOREM 1. *Suppose that I is an ideal in $\Gamma(X, \mathscr{S})$, and that there is a finite subset $\{f_1, \cdots, f_n\} \subset I$ so that for all $x \in X$, there exists a j so that $f_j(x) \neq 0$. Then $I = \Gamma(X, \mathscr{S})$.*

PROOF. Consider the following Koszul complex of sheaves on X:

$$(0) \to \bigwedge^n \mathscr{S}^n \xrightarrow{\delta_n} \bigwedge^{n-1} \mathscr{S}^n \to \cdots$$
(1)
$$\to \bigwedge^k \mathscr{S}^n \xrightarrow{\delta_k} \bigwedge^{k-1} \mathscr{S}^n \to \cdots$$
$$\to \bigwedge^2 \mathscr{S}^n \xrightarrow{\delta_2} \mathscr{S}^n \xrightarrow{\delta_1} \mathscr{S} \to (0).$$

Here $\mathscr{S}^n$ denotes the direct sum of n copies of $\mathscr{S}$, and $\bigwedge^k \mathscr{S}^n$ denotes the kth exterior power of $\mathscr{S}^n$. Also:

$$\delta_1(s_1, \cdots, s_n) = \sum_{i=1}^{n} s_i f_i,$$

and

$$\delta_k(s_{i_1,\cdots,i_k}[i_1] \wedge \cdots \wedge [i_k])$$
$$= (-1)^{j-1}s_{i_1,\cdots,i_k}f_{i_j}[i_1] \wedge \cdots \wedge \widehat{[i_j]} \wedge \cdots \wedge [i_k].$$

Here $\{[i_1] \wedge \cdots \wedge [i_k]\}$ are the basis elements of $\bigwedge^k \mathcal{S}^n$ over $\mathcal{S}$. $\widehat{[i_j]}$ indicates the omission of $[i_j]$. Standard repeated index summation notation has been used.

We now show that sequence (1) is exact. First, an easy computation shows that $\delta_{k-1}\delta_k = 0$, so that sequence (1) is actually a complex. Next, at each point $x \in X$ at least one of the functions $\hat{f}_1, \cdots, \hat{f}_n$ is not zero. Suppose $\hat{f}_l(x) \neq 0$. Then $[f_l]_x \notin m_x$, so $[f_l]_x^{-1} \in \mathcal{S}_x$ exists. Hence given $s \in \mathcal{S}_x$, $s = f_l \cdot (s \cdot [f_l]_x^{-1})$. Thus δ_1 is surjective.

More generally, let $\omega = s_{i_1,\ldots,i_k}[i_1] \wedge \cdots \wedge [i_k] \in (\bigwedge^k \mathcal{S}^n)_x$ and suppose $\delta_k\omega = 0$. Let $\varphi \in (\bigwedge^{k+1} \mathcal{S}^n)_x$ be the element

$$\varphi = (-1)^k[f_l]_x^{-1}s_{i_1,\cdots,i_k}[i_1] \wedge \cdots \wedge [i_k] \wedge [l].$$

Then another computation shows that $\delta_{j+1}(\varphi) = \omega$. Thus sequence (1) is exact.

Each term in (1) is isomorphic to a direct sum of a finite number of copies of $\mathcal{S}$. Since $H^q(X, \mathcal{S}) = (0)$ for all $q \geq 1$, the higher cohomology of each term in (1) vanishes. Hence in sequence (1), if we take global sections, the resulting sequence is exact. In particular, we have

$$(2) \qquad \Gamma(X, \mathcal{S})^n \xrightarrow{\delta_1} \Gamma(X, \mathcal{S}) \to (0)$$

exact; i.e. the map $\delta_1 : \Gamma(X, \mathcal{S})^n \to \Gamma(X, \mathcal{S})$ given by $\delta_1(s_1, \cdots, s_n) = \sum_{j=1}^n s_j f_j$ is surjective. Since $\{f_1, \cdots, f_n\} \subset I$ it follows that $I = \Gamma(X, \mathcal{S})$, and hence the theorem is true.

COROLLARY 1. *Suppose that X is compact. Then for every proper ideal $I \subsetneq \Gamma(X, \mathcal{S})$, there exists an $x \in X$ so that $\hat{f}(x) = 0$ for all $f \in I$.*

PROOF. Suppose the theorem is false. Then for every $x \in X$, there exists $f \in I$ with $\hat{f}(x) \neq 0$. Since $\hat{f}$ is continuous, there is an open neighborhood N of x in X so that $\hat{f}(y) \neq 0$ for all $y \in N$. By compactness, we can find a finite number of these neighborhoods which cover X. Hence there is a finite set $\{f_1, \cdots, f_n\} \subset I$ so that for any $y \in X$, there is an f_j with $\hat{f}_j(y) \neq 0$. By Theorem 1, this implies that $I = \Gamma(X, \mathcal{S})$, so that the corollary is true.

COROLLARY 2. *Let $X \subset C^n$ be a compact set which can be written $X = \bigcap_{i=1}^\infty D_i$, where $D_i \subset C^n$ is a domain of holomorphy. Let $\mathcal{H}(X)$ be the algebra of germs of holomorphic functions on X. Then every maximal ideal of $\mathcal{H}(X)$ is the set of functions vanishing at some fixed point of X.*

PROOF. Let $\mathcal{O}$ denote the sheaf of germs of holomorphic functions on C^n. Then $\mathcal{H}(X) = \Gamma(X, \mathcal{O})$, and for $q \geq 1$, $H^q(X, \mathcal{O}) = \text{dir lim } H^q(D_i, \mathcal{O})$ $= (0)$ by Cartan's Theorem B. (See Gunning and Rossi [1, p. 243].) Since X is compact, the corollary follows from Corollary 1.

COROLLARY 3. *Let $X \subset C^n$ satisfy the same conditions as in Corollary 2. Let $A(X)$ denote the uniform closure of $\mathcal{H}(X)$. Then the maximal ideal space of $A(X)$, viewed as a function algebra, is homeomorphic to X.*

PROOF. It suffices to show that for every algebra homomorphism $\varphi: A(X) \to C$, there exists $x \in X$ so that $\varphi(f) = f(x)$. Let $I = \{f \in \mathcal{H}(X) | \varphi(f) = 0\}$. Then I is a maximal ideal in $\mathcal{H}(X)$ so by Corollary 1, there is $x \in X$ with $\varphi(f) = f(x)$ for all $f \in \mathcal{H}(X)$. Since $\mathcal{H}(X)$ is dense in $A(X)$, and φ is continuous, it follows that $\varphi(f) = f(x)$ for all $f \in A(X)$.

Corollaries 2 and 3 are false without the hypothesis that $X = \bigcap_{i=1}^{\infty} D_i$, where each D_i is a domain of holomorphy, as the standard example $X = \{(z, w) \in C^2 | |z| \leq |w| \leq 1\}$ shows. Also, Corollary 1 is false in general if X is not compact. For example let $X = C$, and $\mathcal{S} = \mathcal{O}$, the sheaf of germs of holomorphic functions on C. Then $H^q(X, \mathcal{S}) = (0)$ for $q \geq 1$ and $\Gamma(X, \mathcal{S})$ is the algebra of entire functions. However if $\{z_n\}$ is an infinite discrete set in C, and I is the ideal of entire functions which vanish at all except some finite subset of $\{z_n\}$, then I is a proper ideal, but the functions in I have no common zero. We can, however, prove the following in the non-compact case:

COROLLARY 4. *Let X and $\mathcal{S}$ satisfy conditions (a), (b), (c). Suppose there are a finite number of global sections $\{f_1, \cdots, f_n\} \subset \Gamma(X, \mathcal{S})$ such that the associated continuous functions on X separate the points of X. Then for every nonzero algebra homomorphism $\varphi: \Gamma(X, \mathcal{S}) \to C$, there exists $x \in X$ so that $\varphi(f) = \hat{f}(x)$ for all $f \in \Gamma(X, \mathcal{S})$.*

PROOF. Let $g_j = f_j - \varphi(f_j)$, for $j = 1, \cdots, n$. Then the functions $\hat{g}_j$, $j = 1, \cdots, n$, still separate the points of X. Let $Z = \{x \in X | \hat{g}_j(x) = 0\}$. Then there is at most one point $x_0 \in Z$. If the corollary is not true, there exists $g_0 \in \Gamma(X, \mathcal{S})$ so that $\varphi(g_0) = 0$ but $\hat{g}_0(x_0) \neq 0$. Then the functions $\{\hat{g}_0, \cdots, \hat{g}_n\}$ have no common zeros. Proceeding as in Theorem 1, we see that every element $f \in \Gamma(X, \mathcal{S})$ can be written $\sum_{i=0}^{n} f_i g_i$ for some $f_i \in \Gamma(X, \mathcal{S})$. But then $\varphi(f) = \sum_{i=0}^{n} \varphi(f_i) \varphi(g_i) = 0$, so φ is the zero homomorphism, a contradiction. Hence the corollary is true.

The following result is now immediate:

COROLLARY 5. *Let X be a Stein space supporting a finite number of holomorphic functions separating points, and let $\mathcal{O}$ be the structure sheaf. Then every nonzero algebra homomorphism $\varphi: \Gamma(X, \mathcal{O}) \to C$ is given by*

evaluation at a point of X. In particular, every algebra homomorphism is continuous if $\Gamma(X, \mathcal{O})$ *is given the usual Fréchet topology.*

Finally, we show that, at least without additional assumptions on the sheaf $\mathcal{S}$, the converse to Corollary 1 is not in general true. Let $X = \{z \in C \,|\, 1 \leqq |z| \leqq 2\}$. Let $\mathcal{S}$ be the sheaf of nonsingular rational functions on X. Then $\Gamma(X, \mathcal{S})$ is the algebra of rational functions on C having no poles on X. It is easy to see that for any proper ideal $I \subset \Gamma(X, \mathcal{S})$, there exists $\lambda \in X$, so that all the functions in I vanish at λ.

Now let $.U_1 = \{z \in X \,|\, \mathrm{Re}(z) < \tfrac{1}{2}\}$, and $U_2 = \{z \in X \,|\, \mathrm{Re}(z) > -\tfrac{1}{2}\}$. Then $\{U_1, U_2\}$ is an open cover of X, and $U_1 \cap U_2$ is the disjoint union of two open sets in X. Define f_{12} in $U_1 \cap U_2$ by setting

$$f_{12}(z) = 0, \quad \text{if } \mathrm{Im}(z) < 0,$$
$$= z, \quad \text{if } \mathrm{Im}(z) > 0.$$

Set $f_{21} = -f_{12}$. Then $\{f_{ij}\}$ is a 1-cocycle with values in $\mathcal{S}$. If $\{f_{ij}\}$ were a coboundary on some refinement, $f_{ij} = F_i - F_j$, then $F_i = F_j$ when $\mathrm{Im}(z) < 0$, so we would have a multiple valued rational function, which is impossible. Hence $H^1(X, \mathcal{S}) \neq (0)$.

REFERENCE

1. R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall Series in Modern Analysis, Prentice-Hall, Englewood Cliffs, N.J., 1965. MR 31 #4927.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706