TWO LIFTING THEOREMS

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Abstract. It is assumed that the measure algebra involved has cardinality \(2^{\aleph_0}\), and it is assumed further that \(2^{\aleph_0}=\aleph_1\). Then liftings exist when the \(\sigma\)-field is not necessarily complete, and strong Borel liftings exist in the locally compact \(\sigma\)-compact metric case.

1. Introduction. Let \((X_0, \mathcal{F}_0, \mu_0)\) be a probability space, let \(B(X_0, \mathcal{F}_0)\) be the Banach algebra of bounded real \(\mathcal{F}_0\) measurable functions on \(X_0\), the norm being \(\|f\| = \sup_{x \in X_0} |f(x)|\), and let \(\mathcal{J}_0 = \{f \in B(X_0, \mathcal{F}_0) : \int f \, d\mu_0 = 0\}\) be the closed ideal of \(\mu_0\)-null functions. The quotient Banach algebra \(B(X_0, \mathcal{F}_0)/\mathcal{J}_0\) may be identified as the familiar \(L_\infty(X_0, \mathcal{F}_0, \mu_0)\); let \(q_0 : B(X_0, \mathcal{F}_0) \to L_\infty(X_0, \mathcal{F}_0, \mu_0)\) be the quotient mapping. A lifting \(\Lambda_0 : L_\infty(X_0, \mathcal{F}_0, \mu_0) \to B(X_0, \mathcal{F}_0)\) is a selection of representative \(\Lambda_0(f + \mathcal{J}_0) \in f + \mathcal{J}_0\) from each equivalence class \(f + \mathcal{J}_0, f \in B(X_0, \mathcal{F}_0)\), in such a way that the representatives constitute a subalgebra of \(B(X_0, \mathcal{F}_0)\); it is required also that \(\Lambda_0(1 + \mathcal{J}_0) = 1\). That is, \(\Lambda_0\) is an algebraic cross section of \(q_0\) which preserves the unit: \(\Lambda_0\) is an algebraic homomorphism, \(q_0 \Lambda_0 = (\text{identity}), \Lambda_0(1 + \mathcal{J}_0) = 1\). The proof in [1] that liftings exist requires \(\mathcal{J}_0\) be complete with respect to \(\mu_0\). We show that \(\mathcal{F}_0\) need not be complete provided: (i) the measure algebra \((\mathcal{F}_0, \mu_0)\) has cardinal \(2^{\aleph_0}\); (ii) \(2^{\aleph_0} = \aleph_1\) (the continuum hypothesis).

Suppose further that \(X_0\) is a topological space, the bounded continuous real functions \(C_b(X_0)\) are \(\mathcal{F}_0\) measurable, and \(\mu_0(U) > 0\) for open \(U \neq \emptyset\). A strong lifting is a lifting \(\Lambda_0\) such that \(\Lambda_0(f + \mathcal{J}_0) = f, f \in C_b(X_0)\). Various sufficient conditions are known [1] for the existence of strong liftings, e.g., \(X_0\) locally compact \(\sigma\)-compact metric. We prove here (with \(2^{\aleph_0} = \aleph_1\)) that strong Borel liftings exist in this last case; that is, strong liftings such that each representative is measurable with respect to the uncompleted \(\sigma\)-field of Borel subsets of locally compact \(\sigma\)-compact metric \(X_0\).

2. Representation spaces. Under the Gel'fand representation; \(B(X_0, \mathcal{F}_0)\) is isometrically algebraically isomorphic to the continuous real functions...
C(W) on a certain compact Hausdorff space W; let \( \iota : C(W) \to B(X_0, \mathcal{F}_0) \) be the inverse isomorphism. This has adjoint \( \iota^* : ba(X_0, \mathcal{F}_0) \to rca(W) \) when we identify the conjugate Banach space \( [B(X_0, \mathcal{F}_0)]^* \) with the space \( ba(X_0, \mathcal{F}_0) \) of bounded finitely additive set functions on \( \sigma \)-field \( \mathcal{F}_0 \), and the conjugate Banach space \( [C(W)]^* \) with the space \( rca(W) \) of regular Borel measures on \( W \). We will assume without essential loss of generality that \( B(X_0, \mathcal{F}_0) \) separates \( X_0 \), i.e., if \( x_1 \neq x_2 \) then \( f(x_1) \neq f(x_2) \) for some \( f \in B(X_0, \mathcal{F}_0) \). Then with \( \delta_x \) the point measure at \( x \), the set \( \{ \iota^* \delta_x : x \in X_0 \} \) constitutes a copy of \( X_0 \) contained as a dense subset of \( W \subset w^*-rca(W) \); we identify \( W \) with the set \( \{ \delta_w : w \in W \} \subset w^*-rca(W) \) whenever convenient.

A bounded real function \( f \) on \( X_0 \subset W \) extends to a member of \( C(W) \) iff \( f \) is represented as the restriction mapping \( g = g|X_0 \), \( g \in C(W) \). Each \( E \in \mathcal{F}_0 \) has closure \( \text{cl}_W E \) which is open, and the correspondence \( E \mapsto \text{cl}_W E \) is 1-1 between \( \mathcal{F}_0 \) and the open closed subsets of \( W \). The \( \sigma \)-field of Baire subsets of \( W \) is generated by \( \{ \text{cl}_W E : E \in \mathcal{F}_0 \} \), and if \( \theta \in ba(X_0, \mathcal{F}_0) \) is given then \( \iota^* \theta \in rca(W) \) is determined on the Baire subsets of \( W \) by \( (\iota^* \theta)(\text{cl}_W E) = \theta(E) \), \( E \in \mathcal{F}_0 \), and then on the Borel sets by regularity.

As a Banach lattice, \( B(X_0, \mathcal{F}_0) \) is boundedly \( \sigma \)-complete (\( = \mathcal{K}_0 \)-reticulated): any countable subset of \( B(X_0, \mathcal{F}_0) \) bounded above has a supremum in \( B(X_0, \mathcal{F}_0) \); the isomorphic \( C(W) \) enjoys the same property. Dually, \( W \) is basically disconnected: disjoint open Baire subsets of \( W \) have disjoint open closures [3, Chapter VII]. Equivalently, the interior \( F^0 \) of any closed Baire set \( F \) is closed, so that a closed Baire set is of the form \( \{ \text{cl}_W E \} \cup N \) with \( E \in \mathcal{F}_0 \) and \( N \) a closed nowhere dense Baire set. The following results from [2] will be used. If \( N \subset W \) is a closed nowhere dense Baire set then \( N = \lim_n \text{cl}_W E_n \) for some sequence \( E_1 = E_2 = \cdots \) in \( \mathcal{F}_0 \) such that \( \lim_n E_n = \emptyset \). If \( \theta \in ba(X_0, \mathcal{F}_0) \) is countably additive on \( \mathcal{F}_0 \) then \( (\iota^* \theta)(N) = 0 \) for every closed nowhere dense Baire set \( N \).

Let \( \mu = \iota^* \mu_0 \in rca(W) \) correspond to \( \mu_0 \), and let \( X \) be the closed support of \( \mu \) in \( W \). The ideal \( \mathcal{I} = \mathcal{I}_i \mathcal{F}_0 \subset C(W) \) which corresponds to \( \mathcal{I} \) is clearly \( \{ f \in C(W) : f(X) = 0 \} \), and the quotient mapping \( q_0 \), isomorphic to the quotient mapping \( C(W) \to C(W)/\mathcal{I} \), is isomorphic to the restriction mapping \( q : C(W) \to C(X) \) given by \( qf = f|X \), \( f \in C(W) \). Space \( X \) is the Gel'fand space of \( L_0(X_0, \mathcal{F}_0, \mu_0) \), and is hyperstonian with \( \mu \) as category measure. That is, \( L_0(X_0, \mathcal{F}_0, \mu_0) \) is isometrically algebraically isomorphic to \( C(X) \), \( X \) is extremally disconnected, and \( \mu(A) = \mu(A^\circ) > 0 \) if \( A^\circ \neq \emptyset \), Borel \( A \subset X \). We denote by \( \mathcal{F} \) the class of open closed subsets of \( X \); the sets \( \{ X \cap \text{cl}_W E : E \in \mathcal{F}_0 \} \) (not necessarily distinct) comprise \( \mathcal{F} \). The measure algebra \( (\mathcal{F}_0, \mu_0) \) is isomorphic to the quotient Boolean algebra \( \mathcal{F}/(\text{nowhere dense sets}) \).
The following relation between the closure operators in $W$ and $X$ is needed in the proof of Theorem 1.

**Lemma 1.** If $U \subseteq W$ is an open Baire set then $X \cap \text{cl}_W U = \text{cl}_X (X \cap U)$.

**Proof.** We have $\Phi = \text{cl}_W U = U \cup N$ with $\Phi$ open closed and $N$ a closed nowhere dense Baire set in $W$. Since $\mu (X \cap N) = 0$ and $\mu$ is category relative to $X$, $X \cap N$ is nowhere dense in $X$. The closure $\Theta = \text{cl}_X (X \cap U) = (X \cap U) \cup N_1$ of open $X \cap U$ is open closed in extremally disconnected $X$. Thus

$$
\Theta = \left[ X \cap (\Phi - N) \right] \cup N_1 = (X \cap \Phi) \Delta [(X \cap N) - N_1],
$$

and since $\Theta$ and $X \cap \Phi$ are each open closed and $X \cap N$ is nowhere dense, $\Theta = X \cap \Phi$. □

3. Partial liftings. To a lifting $\Lambda_0: L_\infty (X_0, \mathcal{F}_0, \mu_0) \rightarrow B(X_0, \mathcal{F}_0)$ there corresponds an algebraic homomorphism $\Lambda: C(X) \rightarrow C(W)$ with the properties $\Lambda 1 = \text{(identity)}$, $\Lambda 1 = 1$; we call $\Lambda$ a lifting also. The adjoint $\Lambda^*: \text{rca} (W) \rightarrow \text{rca} (X)$ restricts to a mapping $\lambda: W \rightarrow X$ which is a retraction of $W$ onto $X$. Conversely, such a retraction determines a lifting according to $(\Lambda f)(w) = f(\lambda w)$, $w \in W, f \in C(X)$.

We denote by $\mathcal{A}$ the set $\mathcal{A} = \{ \alpha \subseteq C(X): \alpha$ is a closed subalgebra of $C(X)$ containing the constants}. Each $\alpha \in \mathcal{A}$ is isometrically algebraically isomorphic to $C(Z_\alpha)$ for a certain compact Hausdorff space $Z_\alpha$. If $j_\alpha : C(Z_\alpha) \rightarrow C(X)$ is the injection onto $\alpha$ then $j_\alpha^*: \text{rca} (X) \rightarrow \text{rca} (Z_\alpha)$ restricts to the quotient mapping $v_\alpha: X \rightarrow Z_\alpha$ associated with $\alpha$, i.e., $(j_\alpha f)(x) = f(v_\alpha x)$, $x \in X, f \in C(Z_\alpha)$.

By a partial lifting $\Lambda_\alpha: \alpha \rightarrow C(W)$ we will mean an algebraic homomorphism defined only on the subalgebra $\alpha$ of $C(X)$ with the properties $q\Lambda_\alpha = \text{(identity)}$, $\Lambda_\alpha 1 = 1$. Equivalent to $\Lambda_\alpha$ is the algebraic homomorphism $\widetilde{\Lambda}_\alpha: C(Z_\alpha) \rightarrow C(W)$ given by $\widetilde{\Lambda}_\alpha = j_\alpha \Lambda_\alpha$ and such that $q\widetilde{\Lambda}_\alpha = j_\alpha$, $\widetilde{\Lambda}_\alpha 1 = 1$. The adjoint $\Lambda_\alpha^*: \text{rca} (W) \rightarrow \text{rca} (Z_\alpha)$ restricts to the partial retraction $\lambda_\alpha: W \rightarrow Z_\alpha$ dual to $\widetilde{\Lambda}_\alpha$ and $\alpha$, i.e., $\lambda_\alpha x = v_\alpha$. Since $\lambda_\alpha$ and $j_\alpha^{-1}$ are isometries, $\Lambda_\alpha = \widetilde{\Lambda}_\alpha j_\alpha^{-1}$ is an isometry.

Let $\mathcal{L}$ denote the family $\mathcal{L} = \{ \Lambda_\alpha: \alpha \in \mathcal{A}$ and $\Lambda_\alpha$ is a partial lifting with domain $\alpha \}$. An order $< \in \mathcal{L}$ is defined by: $\Lambda_\alpha < \Lambda_\beta$ iff $\Lambda_\beta$ extends $\Lambda_\alpha$; that is, $\alpha < \beta$ and $\Lambda_\beta | \alpha = \Lambda_\alpha$.

**Lemma 2.** Any ascending chain in $\mathcal{L}$ has an upper bound in $\mathcal{L}$.

**Proof.** Suppose $\{ \Lambda_\alpha : \alpha \in M \}$ is an ascending chain in $\mathcal{L}$: $M$ is a totally ordered indexing set and $\Lambda_\alpha < \Lambda_\gamma$ for $\mu \leq \nu \in M$. Define subalgebra $\gamma$ of $C(X)$ as $\gamma = \bigcup_\mu \{ \sigma_\mu : \nu \in M \}$, and let $\alpha \in \mathcal{A}$ be the closure in $C(X)$ of $\gamma$. An operator $\Lambda_\gamma : \gamma \rightarrow C(W)$ is defined consistently on $\gamma$ by the family
of its restrictions $\Lambda_{\mu}|_{a_{\mu}} = \Lambda_{a_{\mu}}$, $\mu \in M$, and one verifies easily that $\Lambda_{\mu}$ is an algebraic homomorphism with the properties $q\Lambda_{\mu} = (\text{identity})$, $\Lambda_{\mu}1 = 1$. Since each $\Lambda_{\mu}$, $\mu \in M$, is an isometry, $\Lambda_{a_{\mu}}$ is an isometry, and so extends uniquely by continuity to an operator $\Lambda_{a_{\mu}} : a_{\mu} \to C(W)$. By continuity, $\Lambda_{a_{\mu}}$ is an algebraic homomorphism with the properties $q\Lambda_{a_{\mu}} = (\text{identity})$, $\Lambda_{a_{\mu}}1 = 1$; that is, $\Lambda_{a_{\mu}} \in \mathcal{L}$. Since $\Lambda_{a_{\mu}} \subset \Lambda_{a}$, $\mu \in M$, by construction, $\Lambda_{a}$ is the upper bound sought. □

Suppose $\alpha \in A$ and $X_{1}, X_{2} \in \mathcal{F}$ are given, and let $\beta \in A$ be the algebra generated by $\{\alpha, \chi_{X_{1}}\}$. With $X_{2} = X - X_{1}$, $\beta$ consists of all $f \in C(X)$ of the form $f = f_{1}\chi_{X_{1}} + f_{2}\chi_{X_{2}}$ for some $f_{1}, f_{2} \in \alpha$. We may rewrite this as $f = (f_{1} + \mathcal{F}_{1})\chi_{X_{1}} + (f_{2} + \mathcal{F}_{2})\chi_{X_{2}}$ where $\mathcal{F}_{i} = \{g \in \alpha : g\chi_{X_{i}} = 0\}$, $i = 1, 2$, are closed ideals in $\alpha$; the elements $(f_{i} + \mathcal{F}_{i}) \in \alpha/\mathcal{F}_{i}$, $i = 1, 2$, are then uniquely determined by $f \in \beta$. The space $Z_{\beta}$ associated with $\beta$ is the free union $Z_{\beta} = Z_{\beta_{1}} \cup Z_{\beta_{2}}$ of copies of the subsets $Z_{\beta_{i}} = v_{i}X_{i}$, with $\alpha/\mathcal{F}_{i}$ isomorphic to $C(Z_{\alpha_{i}})$, $i = 1, 2$.

**Lemma 3.** Suppose $\Lambda_{\alpha} \in \mathcal{L}$ and $X_{1}, X_{2} = X - X_{1} \in \mathcal{F}$ are given, and let $\beta \in A$ be the algebra generated by $\{\alpha, \chi_{X_{1}}\}$. If $\Lambda_{\beta} \in \mathcal{L}$ exists such that $\Lambda_{a_{\mu}} \subset \Lambda_{\beta}$ then open closed subsets $W_{1}$ and $W_{2} = W - W_{1}$ of $W$ are determined such that

(i) $\Lambda_{\beta}f = (\Lambda_{a_{\mu}}f_{1})\chi_{W_{1}} + (\Lambda_{a_{\mu}}f_{2})\chi_{W_{2}}$ for $f = (f_{1} + \mathcal{F}_{1})\chi_{X_{1}} + (f_{2} + \mathcal{F}_{2})\chi_{X_{2}} \in \beta$,

(ii) $W_{i} \cap X = X_{i}$, $i = 1, 2$,

(iii) $W_{i} \subset v_{i}X_{i}$, $i = 1, 2$.

Conversely, if open closed $W_{1}$ and $W_{2} = W - W_{1}$ in $W$ are given satisfying (ii) and (iii) then (i) serves to define $\Lambda_{\beta} \in \mathcal{L}$ such that $\Lambda_{a_{\mu}} \subset \Lambda_{\beta}$.

**Proof.** Suppose $\Lambda_{\beta} \in \mathcal{L}$ is given such that $\Lambda_{a_{\mu}} \subset \Lambda_{\beta}$. From $\chi_{X_{1}}^{2} = \chi_{X}$ and $\chi_{X_{2}} = 1 - \chi_{X_{1}}$, and the fact that $\Lambda_{\beta}$ is an algebraic homomorphism such that $\Lambda_{\beta}1 = 1$, we find that $\Lambda_{\beta}\chi_{X_{i}} = \chi_{W_{i}}$, $i = 1, 2$, for open closed subsets $W_{1}$ and $W_{2} = W - W_{1}$ of $W$. Since $\mathcal{F}_{i} \subset \alpha$, $\chi_{X_{i}} \in \beta$, and $\mathcal{F}_{i}\chi_{X_{i}} = 0$, $i = 1, 2$, we must have

$\Lambda_{\beta}(\mathcal{F}_{i}\chi_{X_{i}}) = (\Lambda_{a_{\mu}}\mathcal{F}_{i})(\Lambda_{\beta}\chi_{X_{i}}) = (\Lambda_{a_{\mu}}\mathcal{F}_{i})\chi_{W_{i}} = 0$, $i = 1, 2$;

this is condition (iii). The property $q\Lambda_{a_{\mu}} = (\text{identity})$ gives condition (ii). The converse arguments are similar. □

**Theorem 1.** Suppose $\Lambda_{\alpha} \in \mathcal{L}$ and $X_{1}, X_{2} \in \mathcal{F}$ are given, and let $\beta \in A$ be the subalgebra generated by $\{\alpha, \chi_{X_{1}}\}$. If $Z_{\alpha}$ is metrizable then partial liftings $\Lambda_{\beta}$ exist which extend $\Lambda_{\alpha}$.

**Proof.** The conditions $W_{1} \subset v_{i}^{-1}X_{i}$, $i = 1, 2$, of Lemma 3 are equivalent to $W_{i} \supset v_{i}^{-1}U_{i}$, $i = 1, 2$, where $U_{i} = Z_{\alpha} - v_{i}X_{i}$, $i = 1, 2$, are disjoint open subsets of $Z_{\alpha}$. If $Z_{\alpha}$ is metrizable the Borel sets are Baire sets, $U_{1}$ and $U_{2}$ are disjoint open Baire sets in $Z_{\alpha}$, whence $v_{i}^{-1}U_{i}$ and $v_{i}^{-1}U_{2}$ are disjoint...
open Baire sets in $W$. Using the fact that $W$ is basically disconnected, we have that $\Phi_1 = \text{cl}_W(\lambda^{-1}U_1)$ and $\Phi_2 = \text{cl}_W(\lambda^{-1}U_2)$ are disjoint open closed subsets of $W$.

By Lemma 1,

$$X \cap \Phi_1 = \text{cl}_X(X \cap \lambda^{-1}U_1) = \text{cl}_X(v_\lambda^{-1}U_1) = \text{cl}_X(X - v_\lambda^{-1}v_\alpha X_1) \subseteq X_1,$$

since $X - v_\lambda^{-1}v_\alpha X_2 \subseteq X_1$ and $X_1$ is closed; similarly, $\Phi_2 \cap X \subseteq X_2$.

Let $\Gamma_1$ and $\Gamma_2 = W - \Gamma_1$ be any open closed subsets of $W$ such that $X \cap \Gamma_i = X_i$, $i = 1, 2$. With $\Theta = W - (\Phi_1 \cup \Phi_2)$ open closed, define open closed $W_i$ and $W_2$ by $W_i = \Phi_i \cup (\Theta \cap \Gamma_i)$, $i = 1, 2$. It is clear that $W_2 = W - W_1$. Since $W_i \cap X = (\Phi_i \cap X) \cup (\Theta \cap X_i) \subseteq X_i$ and $\{W_1, W_2\}$, $\{X_1, X_2\}$ are partitions, we have necessarily $W_i \cap X = X_i$, $i = 1, 2$. Conditions (ii) and (iii) of Lemma 3 being satisfied, (i) gives the extension sought. □

4. The Lifting theorems. The cardinal of the measure algebra $(\mathcal{F}_0, \mu_0)$ is either finite or at least $2^{\aleph_0}$; we assume from now on that the cardinal is $2^{\aleph_0}$. We assume further that $2^{\aleph_0} = \aleph_1$, and we let $\{F_v : v < \aleph_1\}$ be a well ordering of the elements of $\mathcal{F}$.

**Theorem 2 (Incomplete lifting theorem).** If the measure algebra $(\mathcal{F}_0, \mu_0)$ has cardinal $2^{\aleph_0} = \aleph_1$ then liftings $\Lambda_0 : L^{\infty}(X_0, \mathcal{F}_0, \mu_0) \to B(X_0, \mathcal{F}_0)$ exist.

**Proof.** The parts of the transfinite induction are:

(i) $\sigma_0$ is the constants, $Z_{\sigma_0}$ is a singleton, $\Lambda_{\sigma_0} = 1$.

(ii) For $v < \aleph_1$, a successor ordinal, suppose $\{\Lambda_{\sigma v} : \gamma < v\}$ is an ascending chain in $\mathcal{L}$ such that each $Z_{\sigma v}$, $\gamma < v$, is metrizable; in particular, $Z_{\sigma v-1}$ is metrizable. Let $\alpha_v$ be the algebra generated by $\{x_{v-1}, x_{F v-1}\}$, and let $\Lambda_{\sigma v}$ be the partial lifting provided by Theorem 1. It is clear that $Z_{\sigma v}$ is metrizable, so that $\{\Lambda_{\sigma v} : \gamma < v + 1\}$ is an ascending chain in $\mathcal{L}$ such that each $Z_{\sigma v}$ is metrizable, $\gamma < v + 1$.

(iii) For $v < \aleph_1$, a limit ordinal, suppose $\{\Lambda_{\sigma v} : \gamma < v\}$ is an ascending chain in $\mathcal{L}$ such that each $Z_{\sigma v}$, $\gamma < v$, is metrizable. Lemma 2 provides $\Lambda_{\sigma v}$ on $\alpha_v = \text{cl}_{C(X)} \cup_{\gamma < v} \alpha_v$ such that $\{\Lambda_{\sigma v} : \gamma < v + 1\}$ is an ascending chain in $\mathcal{L}$. If $\sigma_v \subset \alpha_v$ is a countable set dense in $\alpha_v$, $\gamma < v$, then $\alpha_v \subset \sigma_v$ is a countable set dense in $\alpha_v$, so that $\Lambda_{\sigma v}$ is metrizable.

By transfinite induction, there exists an ascending chain $\{\Lambda_{\sigma v} : \gamma < \aleph_1\}$, and Lemma 2 provides an ascending chain $\{\Lambda_{\sigma v} : \gamma \leq \aleph_1\}$. The algebra $\alpha_{\aleph_1} = \cup_{\gamma < \aleph_1} \alpha_v \in \mathcal{A}$ contains every $\chi_{F v}$, $v < \aleph_1$, and so is all of $C(X)$. Thus the partial lifting $\Lambda_{\sigma_{\aleph_1}}$ is a lifting. □

**Theorem 3 (Strong Borel lifting theorem).** Let $X_0$ be a locally compact $\sigma$-compact metric space, let $\mathcal{F}_0$ be the Borel subsets of $X_0$, let $\mu_0$ be strictly positive on nonempty open sets, and assume $2^{\aleph_0} = \aleph_1$. Then
liftings $\Lambda_0 : L_\infty(X_0, \mathcal{F}_0, \mu_0) \to B(X_0, \mathcal{F}_0)$ exist such that $\Lambda_0(f + J_0) = f$, $f \in C_b(X_0)$.

**Proof.** With $C_0(X_0)$ the continuous real functions vanishing at infinity, let $A_0 \subseteq C_b(X_0)$ be the algebra generated by $\{C_0(X_0), 1\}$; $A_0$ is isometrically algebraically isomorphic to $C(Z_{\alpha})$ where $Z_{\alpha} = X_0 \cup \{\alpha\}$ is the one point compactification of $X_0$ if $X_0$ is noncompact, or $Z_{\alpha} = X_0$ if $X_0$ is compact. The assumption that $\mu_0$ is strictly positive on nonempty open sets implies that for each $f \in A_0$, $f$ is the unique continuous function in the class $f^+J_0 \in L_\infty(X_0, \mathcal{F}_0, \mu_0)$. Equivalently, a partial lifting $\Lambda_{\alpha} : a_0 \to C(W)$ of the subalgebra $a_0 = \alpha^{-1}A_0 \subseteq C(X)$ is determined such that $\Lambda_{\alpha}(a_0) = f$, $f \in A_0$. The space $Z_{\alpha}$ associated with $\alpha_0$ is the one defined above, and the assumption that $X_0$ is $\sigma$-compact implies that $Z_{\alpha}$ is metrizable.

We now apply transfinite induction; parts (ii) and (iii) are as in the proof of Theorem 2, but part (i) is: $\alpha_0 = \alpha^{-1}A_0$, $Z_{\alpha_0}$ and $\Lambda_{\alpha_0}$ as just described. We obtain a lifting $\Lambda : C(X) \to C(W)$ such that $\Lambda_{\alpha} < \Lambda$; the isomorphic $\Lambda_0 : L_\infty(X_0, \mathcal{F}_0, \mu_0) \to B(X_0, \mathcal{F}_0)$ is such that $\Lambda_0(f + J_0) = f$, $f \in A_0$.

If $X_0$ is compact we are done; suppose $X_0$ is noncompact. Since $X_0$ is assumed to be $\sigma$-compact, there exists $h \in C_0(X_0)$ such that $h(x) > 0$, $x \in X_0$. From

$$
\Lambda_0(hf + J_0) = [\Lambda_0(h + J_0)][\Lambda_0(f + J_0)], \quad f \in B(X_0, \mathcal{F}_0),
$$

and $\Lambda_0(h + J_0) = h > 0$ we have

$$
\Lambda_0(f + J_0) = h^{-1}\Lambda_0(hf + J_0), \quad f \in B(X_0, \mathcal{F}_0).
$$

If $f \in C_b(X_0)$ then $hf \in C_0(X_0)$ and $\Lambda_0(hf + J_0) = hf$, giving $\Lambda_0(f + J_0) = f$, $f \in C_b(X_0)$. That is, $\Lambda_0$ is a strong Borel lifting. \( \square \)

We conclude with the following remarks. In the proof of the lifting theorem given in [1] it is required that the subalgebras $\alpha$ involved in the partial liftings be boundedly complete; that is, the $Z_{\alpha}$ are extremally disconnected. In the induction step corresponding to Theorem 1 of the present paper the sets $U_1$, $U_2 \subseteq Z_{\alpha}$ have closures in $Z_{\alpha}$ which are disjoint and open closed, hence Baire, and these closures can replace $U_1$, $U_2$ in the argument. The induction step corresponding to Lemma 2 becomes much more difficult, however. The partial liftings $\Lambda_{\alpha}$, $\nu \in M$, must be extended not only to our $\alpha = \text{cl}_{C(X)}\{\bigcup \{x_\nu : \nu \in M\}\}$ (this is the elementary $L_\infty$ martingale theorem given above) but to the boundedly complete algebra generated by $\alpha$; this requires the completeness of $\mathcal{F}_0$ with respect to $\mu_0$ [1, Theorem IV. 2].

**Added in proof.** Theorem 2 of the present paper, but not Theorem 3, can be derived from the results of [4].
REFERENCES


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