THE DENSITY CHARACTER OF FUNCTION SPACES

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ABSTRACT. For topologies between the pointwise topology and the compact-open topology, the density character of $C(X)$ (and, for certain spaces $Z$, $C(X,Z)$) is described in terms of a cardinal invariant of $X$. The Hewitt-Pondiczery theorem on the density character of product spaces follows as a corollary.

1. Description. Except in Corollary 2, all hypothesized spaces are assumed to be completely regular Hausdorff. The set of continuous functions from $X$ to $Z$ is denoted by $C(X,Z)$ or, when $Z = R$, by $C(X)$. The density character, $\delta X$, of a space $X$ is the least cardinality of a dense subset of $X$ and the weight, $wX$, of $X$ is the least cardinality of an open basis of $X$. We define the weak weight, $wwX$, of $X$ to be the least of the cardinals $wY$ for a continuous one-to-one image of $X$.

THEOREM 1. Let $X$ be any infinite space and let $C(X)$ have any topology between the pointwise topology and the compact-open topology. Then $\delta C(X) = wwX$.

All proofs will be given in the next section. Our remaining results allow Theorem 1 to be applied to yield information about $\delta C(X, Z)$ for suitable spaces $Z$.

LEMMA. Let $C(X, Z)$ and $C(X, Z^*)$ both have either the topology of uniform convergence on members of some fixed cover of $X$ or the set-open topology generated by such a cover. If $Z$ is a retract of $Z^*$ then $\delta C(X, Z) \leq \delta C(X, Z^*)$.

PROPOSITION. For any topologies between the pointwise topology and the compact-open topology:

(i) For any space $Z$, $\delta Z \leq \delta C(X, Z)$.

(ii) If $Z$ contains a nondegenerate arc, then $\delta C(X) \leq \delta C(X, Z)$.

(iii) If for each finite subset $F$ of $X$ there exists a function $f$ in $C(X, Z)$ such that $F$ and $f(F)$ have the same cardinality, then $\delta C(X) \leq \delta C(X, Z) \cdot \delta C(Z)$.

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Theorem 2. Suppose \( Z \) contains more than one point and is a retract of a convex subset of some locally convex linear space. For any topologies between the pointwise topology and the compact-open topology, \( \delta C(X, Z) = \delta C(X) \cdot \delta Z \).

Corollary 1. For \( Z \) as in Theorem 2 and any topology between the pointwise topology and the compact-open topology, \( C(X, Z) \) is separable if and only if \( Z \) is separable and some continuous one-to-one image of \( X \) is separable metrizable.

Under the same hypotheses on a separable space \( Z \), Vidossich [6] shows that \( C(X, Z) \) is separable if and only if \( X \) is submetrizable (has a continuous one-to-one metrizable image) with \( \delta X \leq c \), which is equivalent to being submetrizable and of cardinality less than or equal to \( c \). Combined with Corollary 1 this yields:

Corollary 2. On a set of cardinality less than or equal to \( c \) each metrizable topology contains a separable metrizable topology.

Theorem 3. For the topology of pointwise convergence, if \( Z \) is arcwise connected or if \( X \) is a P-space then \( \delta C(X, Z) \leq \delta C(X) \cdot \delta Z \).

Corollary 3 (Hewitt [2], Pondiczery [3]). Let \( n \) be an infinite cardinal. With no separation axioms assumed, let \( X = \prod_{a \in A} X_a \) where each \( X_a \) has \( \delta X_a \leq n \). If \( \operatorname{card} A \leq 2^n \), then \( \delta X \leq n \).

Turning to other topologies on \( C(X) \) we note that the compact-open topology arises in Theorem 1 only because we need the Stone-Weierstrass Theorem (that a subalgebra is dense if it contains the constants and separates points). When dealing with a subalgebra generated by a set of cardinality \( n \) which separates points, compactness can be weakened to weak-\( n \)-\( \mathcal{K}_0 \)-compactness (each \( n \)-fold open cover has a finite subcollection whose union is dense—such spaces are studied, for example, in [4]) so if \( \omega \omega X = n \), then in the set-open topology generated by all weakly-\( n \)-\( \mathcal{K}_0 \)-compact subsets of \( X \), \( \delta C(X) \leq n \). In case \( \omega \omega X = \mathfrak{K}_0 \), an even stronger statement holds: \( C(X) \) is separable in the set-open topology generated by all relatively pseudocompact subsets of \( X \). (A set \( S \subseteq X \) is relatively pseudocompact if each function in \( C(X) \) is bounded on \( S \); weak-\( \mathcal{K}_0 \)-\( \mathcal{K}_0 \)-compactness is equivalent to pseudocompactness.)

Of course the most interesting topology on \( C(X) \) larger than the compact-open topology is the topology of uniform convergence, and for this topology it is known that \( \delta C(X) = \omega \beta X \): That \( \delta C^*(X) = \omega \beta X \) is established by Smirnov in [5] by a proof similar to the proof of Theorem 1 and the more difficult result \( \delta C^*(X) = \delta C(X) \) is shown by Comfort and Hager in [1].
Since the largest set-open topology on $C(X)$ is the topology of uniform convergence if (and only if) $X$ is pseudocompact, it follows by the Comfort-Hager-Smirnov Theorem, the comments above, and the fact that $\beta X$ is separable only if $X$ is compact, that if $X$ is compact separable metric, no strictly larger topology on $X$ is pseudocompact.

2. The proofs.

Proof of Theorem 1. We first show that $\delta C(X) \leq \omega \omega X$; since larger topologies have "larger" density characters, it suffices to consider the case where $C(X)$ has the compact-open topology. Let $\mathfrak{B}$ be a base for some topology contained in the topology of $X$ with $\text{card} (\mathfrak{B}) = \omega \omega X$ and form a set of functions $D \subseteq C(X, I)$ by choosing, for each pair $(U, V)$ in $\mathfrak{B} \times \mathfrak{B}$ with $U \subseteq V$, a function which maps $U$ to 0 and $X \setminus V$ to 1, if such a function exists. Clearly $D$ separates points of $X$, so by the Stone-Weierstrass Theorem (for example, [7, problem 44B]) the algebra generated by $D$ and the constant functions is dense in $C(X)$. Hence the set, $D^*$, of all finite linear combinations, with rational coefficients, of members of $D$ or rational constant functions is dense in $C(X)$. Since $\text{card} D^* = \text{card} D = \omega \omega X$, this shows that $\delta C(X) \leq \omega \omega X$.

For the remaining relation, $\omega \omega X \leq \delta C(X)$, it suffices to consider $C(X)$ in the pointwise topology. Let $D \subseteq C(X)$ be dense with $\text{card} (D) = \delta C(X)$ and let $e: X \to \mathbb{R}^D$ be the embedding map, $e(x)_d = d(x)$. Since $D$ is dense, it separates points, so $e$ is one-to-one. Since each subspace of $\mathbb{R}^D$ has weight $\leq \text{card} D = \delta C(X)$, it follows that $\omega \omega X \leq \delta C(X)$.

Proof of the lemma. For $D \subseteq C(X, Z^*)$ dense and $r: Z^* \to Z$ a retraction, $\{r \circ d: d \in D\}$ is dense in $C(X, Z)$.

Proof of the proposition. (i) Identifying $Z$ with the constant functions and choosing $z \in Z$, the function which carries $/ \to f(z)$ is a retraction of $C(X, Z)$ to $Z$. Thus, by the lemma, $\delta Z \leq \delta C(X, Z)$.

(ii) By the lemma $\delta C(X, I) \leq \delta C(X, Z)$ and since

$$C^*(X) = \bigcup_n C(X, [-n, n])$$

is dense in $C(X)$, $\delta C(X) = \delta C(X, I)$. Therefore $\delta C(X) \leq \delta C(X, Z)$.

(iii) To see that $\delta C(X) \leq \delta C(X, Z) \cdot \delta C(Z)$ it suffices to consider the pointwise topologies, since by Theorem 1, $\delta C(X)$ does not change, and $\delta C(X, Z)$ can only be larger. Let $D_1$ and $D_2$ be dense in $C(X, Z)$ and $C(Z)$ respectively with $\text{card} D_1 = \delta C(X, Z)$, $\text{card} D_2 = \delta C(Z)$, and set $D = \{g \circ f: f \in D_1, g \in D_2\}$. We show that $D$ is dense in $C(X)$. Given distinct points $x_1, \cdots, x_n$ in $X$ and open subsets $U_1, \cdots, U_n$ of $R$ with $\cap_{i=1}^n N(x_i, U_i) \neq \emptyset$ (where $N(x, U) = \{f: f(x) \in U\}$), there exists a function $h$ in $C(X, Z)$ such that the points $h(x_i)$ are all distinct. Choosing disjoint open neighborhoods $V_i$ of $h(x_i)$, $\cap_{i=1}^n N(x_i, V_i)$ is thus a nonempty open.
set, so it contains a function $f$ in $D_1$. Since $\bigcap_{i=1}^n N(x_i, U_i) \neq \varnothing$, it contains a function which maps the points $x_i$ to distinct reals and hence, shrinking the $U_i$ if necessary, we may suppose that they are all disjoint. Thus $\bigcap_{i=1}^n N(f(x_i), U_i)$ is not empty, so it contains a function $g$ in $D_2$. But now $g \cdot f$ is in $\bigcap_{i=1}^n N(x_i, U_i)$, as desired.

**Proof of Theorem 2.** Since $Z$ must contain a nondegenerate arc, the relation $\delta C(X, Z) \supseteq \delta C(X) \cdot \delta Z$ follows from the first two parts of the proposition. For the reverse inequality it suffices to consider the case where $Z$ is a convex subset of a locally convex linear space $Z^*$ which contains the origin and where $C(X, Z)$ and $C(X)$ bear the compact-open topologies. Let $D'$ be a dense subset of $C(X, I)$ of cardinality $\delta C(X)$ (that $\delta C(X, I) = \delta C(X)$ follows from the lemma and the second part of the proposition) and let $D''$ be the set of convex combinations, with rational coefficients, of members of $D'$. For $g$ and $g'$ in $D''$ and for $r$ and $s$ rational with $0 \leq r < s \leq 1$ choose $h(g, g', r, s)$ in $C(X, I)$ such that for $h = h(g, g', r, s)$, $h(x) \leq g'(x)$ for all $x$ in $X$, $h(x) = g'(x)$ if $x$ is in $g^{-1}([0, r])$, and $h(x) = 0$ if $x$ is in $g^{-1}([s, 1])$. Finally let $D_1$ be the set of all functions $h(g, g', r, s)$ so chosen and note that the cardinality of $D_1$ is $\delta C(X)$. Now let $D_2$ be a dense subset of $Z$ of cardinality $\delta Z$ and, identifying $z$ in $D_2$ with the constant function on $X$ whose value is $z$, set

$$D = \left\{ \sum_{i=1}^m f_i z_i : f_i \in D_1, z_i \in D_2 \right\}.$$ 

Since the cardinality of $D$ is $\delta C(X) \cdot \delta Z$, it suffices to show that $D \cap C(X, Z)$ is dense in $C(X, Z)$.

Let $K$ be a compact subset of $X$, let $\rho$ be a continuous seminorm for $Z^*$, let $\epsilon$ be a positive rational less than 1 and choose $f$ in $C(X, Z)$. Let $S = \{ z \in Z : \rho(z) < \epsilon/2 \}$, cover $K$ with the sets $U(x) = f^{-1}(f(x) + S)$ for $f(x)$ in $D_2$ and let $U(x_1), \ldots, U(x_n)$ be a finite subcover. Let $\Phi_1, \ldots, \Phi_n$ be a partition of unity subordinate to this cover, let $M$ be a positive integer greater than each of the numbers $2n \rho(f(x_i))$, $1 \leq i \leq n$, and choose functions $f_i$ in $D'$ such that for each $x$ in $K$, $\Phi_i(x) - \epsilon/M < f_i(x) < \Phi_i(x) - \epsilon/2M$. Set $g = (1/n) \sum_{i=1}^n f_i$, $r = 1/n - \epsilon/2M$ and $s = 1/n$ and let $h_i = h(g, f_i, r, s)$, $1 \leq i \leq n$. Note that for $x$ in $K$,

$$g(x) = \left( \frac{1}{n} \right) \left( \sum_{i=1}^n f_i(x) \right) \leq \left( \frac{1}{n} \right) \left( \sum_{i=1}^n \Phi_i(x) - \frac{n \epsilon}{2M} \right) = r,$$

so $h_i(x) = f_i(x)$. Furthermore, for any $x$, $\sum_{i=1}^n h_i(x) \leq \sum_{i=1}^n f_i(x)$ and $\sum_{i=1}^n h_i(x) = 0$ if $\sum_{i=1}^n f_i(x) \geq 1$, so $\sum_{i=1}^n h_i(x) \leq 1$. Therefore the function $h = \sum_{i=1}^n h_i f(x_i)$ is in $D \cap C(X, Z)$. We complete the proof by showing that
\[ \rho(h(x) - f(x)) < \varepsilon \text{ for } x \text{ in } K. \] For such \( x, \)
\[
\rho(h(x) - f(x)) = \rho \left( \sum_{i=1}^{n} h_i(x) f(x_i) - \sum_{i=1}^{n} \Phi_i(x) f(x) \right)
\]
\[
= \rho \left( \sum_{i=1}^{n} (h_i(x) - \Phi_i(x)) f(x_i) + \sum_{i=1}^{n} \Phi_i(x) f(x_i) - f(x) \right)
\]
\[
\leq \rho \left( \sum_{i=1}^{n} |f(x_i)| |\Phi_i(x) - f(x_i)| + \left( \sum_{i=1}^{n} \Phi_i(x) \right) (\varepsilon/2) \right).
\]
\[
\leq \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

**Proof of Theorem 3.** We first suppose that \( Z \) is arcwise connected. Let \( D' \) be a dense subset of \( Z \) of cardinality \( \delta Z \) and for each finite subset \( F \) of \( D' \) choose an arcwise connected subspace \( X_F \) of \( Z \) which contains \( F \), is the union of finitely many arcs, and which contains no closed loops. (To construct such a subspace, suppose inductively that \( Y_k \) has been constructed so as to satisfy these conditions and contain \( k \) points of \( F \), let \( z \) be a point in \( F \setminus Y_k \), let \( f \) be an arc with \( f(0) \) in \( Y \) and \( f(1) = z \) and let \( r \) be the largest number such that \( f(r) \) is in \( Y_k \). The space \( Y_{k+1} = Y_k \cup f([r, 1]) \) has the required properties and contains \( k+1 \) points of \( F \).) The space \( X_F \) is a retract of \( R^2 \), so \( \delta C(X, X_F) \subseteq \delta C(X) \) and hence \( C(X, X_F) \) contains a dense subset \( D_F \) of cardinality at most \( \delta C(X) \). Set \( D = \bigcup D_F \) is a finite subset of \( D' \) and note that the cardinality of \( D \) is at most \( \delta C(X) \cdot \delta Z \).

Observe that each \( X_F \) can be expressed as \( h_F(I) \) for some continuous function \( h_F \). Hence if \( F' \subseteq X \) is finite and \( f:F' \rightarrow F' \subseteq D' \) is given, then \( f \) has a continuous extension \( f^*:X \rightarrow X_F \). (Indeed, for \( g:F' \rightarrow I \) with each \( g(x) \) in \( h_F^{-1}(f(x)) \) and \( g^*:X \rightarrow I \) a continuous extension of \( g \), the function \( f^* = h_F \circ g^* \) is such an extension.) It is now easy to show that \( D \) is dense.

Given distinct points \( x_1, \ldots, x_n \) of \( X \) and nonempty open subsets \( U_1, \ldots, U_n \) of \( Z \), choose points \( z_i \) in \( F' \cap U_i \), set \( F = \{z_1, \ldots, z_n\} \), let \( f \) be the function which carries \( x_i \) to \( z_i \) and let \( f^*:X \rightarrow X_F \) be a continuous extension of \( f \). Then \( f^* \) is in \( \bigcap_{i=1}^{n} N(x_i, U_i) \), so \( \bigcap_{i=1}^{n} N(x_i, U_i) \) is not empty. It follows that \( D_F \), and hence \( D \), meets \( \bigcap_{i=1}^{n} N(x_i, U_i) \), as desired.

Now suppose that \( X \) is a \( P \)-space. Let \( Z^* \) be the quotient of \( I \times Z \) formed by identifying \( \{1\} \times Z \) to a point \( v \), choose a point \( z_0 \) in \( Z \) and let \( q:Z^* \rightarrow Z \) be the map which carries \( v \) to \( z_0 \) and points \( (r, z) \) to \( z \). Since \( Z^* \) is arcwise connected, \( C(X, Z^*) \) contains a dense subset \( D' \) of cardinality at most \( \delta C(X) \cdot \delta Z \). Note that if \( f:X \rightarrow Z^* \) is continuous then \( q \circ f \) is continuous since \( f^{-1}(v) \), being a closed \( G_\delta \), is both closed and open. Thus \( D = q \circ f : f \in D' \) is a subset of \( C(X, Z) \) of cardinality at most \( \delta C(X) \cdot \delta Z \). Since \( D \) is clearly dense in the pointwise topology, the proof is complete.
Proof of Corollary 3. Let $X = \prod_{a \in A} X_a$ where $|A| = 2^n$ and where, we may suppose, $A$ is discrete and each $X_a$ contains a dense subset $D_a$ of cardinality $n$. Let $Z_a = Z$, the discrete space of cardinality $n$, and let $D'$ be a dense subset of $\prod_{a \in A} Z_a = C(A, Z)$ of cardinality at most $\delta C(A) \cdot \delta Z = w \omega A \cdot \delta Z = n \cdot n = n$. (Since $A$ is discrete, $w \omega A$ is the least cardinal $m$ such that $|A| \leq \text{card } R^m = 2^m$, so $w \omega A = n$.) Let $i_a : Z_a \rightarrow D_a$ be one-to-one onto and set $D = (\prod_{a \in A} i_a)(D')$. Since the projection of $D'$ onto any finite subproduct must be onto, it is clear that $D$ is dense, as desired.

References

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