PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 1, January 1974

THE VOLUME OF A REGION DEFINED BY POLYNOMIAL INEQUALITIES

O. S. ROTHAUS[1]

ABSTRACT. Let $P(x)$ be a polynomial on $\boldsymbol{R}^n$ with nonnegative coefficients. We develop a simple necessary and sufficient condition that the set $S=\{x\in\boldsymbol{R}^n|x_i\geqq 0,\ P(x)\leqq 1\}$ shall have finite volume. A corresponding result where $P(x)$ is replaced by a collection of polynomials is an easy corollary. Finally, the necessary and sufficient conditions for the special case that P is a product of linear forms is also given.

Let $P(x)$ be a polynomial on $\boldsymbol{R}^n$ with nonnegative coefficients, and without constant term (to avoid trivial complications).

$$P(x)=\sum_{v=1}^{k} r_v x_1^{c_{v(1)}} x_2^{c_{v(2)}} \cdots x_n^{c_{v(n)}}, \qquad r_v>0.$$

The vectors $c_v=(c_{v(1)}, c_{v(2)}, \cdots, c_{v(n)})$ are called the exponents of P. Let C be the closed convex cone in $\boldsymbol{R}^n$ generated by the c_v, i.e., the elements of C are all linear combinations $p_1c_1+p_2c_2+\cdots+p_kc_k$ with $p_i\geqq 0$. Let $\langle\ ,\ \rangle$ be the usual inner product in $\boldsymbol{R}^n$, and let C^* be the dual cone to C with respect to this scalar product; i.e., C^* is the set of $y\in\boldsymbol{R}^n$ such that $\langle y, x\rangle\geqq 0\ \forall x\in C$. Note that C^* contains the first 2^n-gant in $\boldsymbol{R}^n$, so C^* has nonempty interior.

There are several well-known features of the above situation, which it is easy to establish using separation properties of convex sets. Thus if b is not an interior point of C, there exists $d\in C^*$, $d\neq 0$, such that $\langle d, b\rangle\leqq 0$. While if $b\neq 0$ is an interior point of C, then there exists a positive constant p such that $\langle d, b\rangle\geqq p\langle d, d\rangle^{1/2}\ \forall d\in C^*$, as an easy compactness argument shows. Then we have

THEOREM 1. *The set* $S=\{x|x_i\geqq 0,\ P(x)\leqq 1\}$ *is of finite volume if and only if the vector* $m=(1, 1, \cdots, 1)$ *is an interior point of* C. (*In particular,* C *must have a nonempty interior.*)

Received by the editors February 26, 1973.

AMS (MOS) subject classifications (1970). Primary 52A20; Secondary 10E05, 10E15.

[1] Work partially supported by NSF GP8129.

PROOF. (S is convex, but we do not need this fact.)

$$\operatorname{Vol} S = \int_{x_i \geqq 0, P(x) \leqq 1} dx = \int_{P(e^{-u}) \leqq 1} e^{-\langle m,u \rangle} \, du.$$

Now pick a vector y such that $\langle c_v, y \rangle \geqq \log kr_v$. Then if $u \in C^* + y$,

$$P(e^{-u}) = \sum_{v=1}^{k} r_v e^{-\langle c_v, u \rangle} \leqq \sum_v r_v \frac{1}{kr_v} = 1.$$

So $\{u \in \boldsymbol{R}^n | P(e^{-u}) \leqq 1\} \subset C^* + y$.

Also pick a vector w such that $\langle c_v, w \rangle \leqq \log r_v$. Then if $P(e^{-u}) \leqq 1$, we must have $r_v e^{-\langle c_v, u \rangle} \leqq 1$, which implies that $\langle c_v, u \rangle \geqq \log r_v$, which implies that $\langle c_v, u-w \rangle \geqq 0$, i.e., $u \in w + C^*$.

Thus the set $\{u \in \boldsymbol{R}^n | P(e^{-u}) \leqq 1\}$ is contained in some translate of C^*, and contains a second translate. It follows that Vol S is finite if and only if $\int_{C^*} e^{-\langle m,u \rangle} du$ is finite. But if m is an interior point of C, then $\langle m, u \rangle \geqq p \langle u, u \rangle^{1/2}$ for $u \in C^*$, and the integral is obviously finite. While if m is not an interior point, it is easy to see that the above integral diverges, completing the proof.

COROLLARY. *Let $P_1, P_2, \cdots, P_r$ be polynomials on $\boldsymbol{R}^n$ with nonnegative coefficients. The set*

$$S = \{x \mid x_i \geqq 0, P_1(x) \leqq 1, P_2(x) \leqq 1, \cdots, P_r(x) \leqq 1\}$$

is of finite volume if and only if $m = (1, 1, \cdots, 1)$ is an interior point of the cone generated by the exponents of all the polynomials P_i.

For if $x \in S$, then $r^{-1}P_1(x) + r^{-1}P_2(x) + \cdots + r^{-1}P_r(x) \leqq 1$, while if $P_1(x) + P_2(x) + \cdots + P_r(x) \leqq 1$, $x \in S$.

Next, we apply the above theorem to the case when $P(x)$ is a product of linear forms on $\boldsymbol{R}^n$.

$$P(x) = \prod_{v=1}^{k} (a_{v(1)}x_1 + a_{v(2)}x_2 + \cdots + a_{v(n)}x_n),$$

each linear form having nonnegative coefficients not all zero. Let U be a subset of $\{1, 2, \cdots, n\}$. We say that the support of the linear form $a_1x_1 + a_2x_2 + \cdots + a_nx_n$ is U if $a_i \neq 0$ for $i \in U$, and $a_i = 0$ for $i \notin U$. For any subset U, let $N(U)$ be the number of linear forms in product for $P(x)$ whose support is contained in U. Then we have:

THEOREM 2. Vol *S is finite if and only if for every proper subset U, we have $N(U)/\text{card}\, U < k/n$.*

To prove the "if" part, let $u=(u_1, u_2, \cdots, u_n) \in C^*$, and suppose without loss of generality that $u_1 \geqq u_2 \geqq \cdots \geqq u_n$. For $1 \leqq r \leqq n$, put $N_r = N(\{1, 2, \cdots, r\})$. Then the vector $c=(N_1, N_2-N_1, N_3-N_2, \cdots, N_n-N_{n-1})$ is an exponent of P.

Hence

$$\begin{aligned}\langle c, u\rangle &= N_1(u_1-u_2)+N_2(u_2-u_3)+\cdots+N_{n-1}(u_{n-1}-u_n)+ku_n \\ &\leqq (k/n)(u_1+u_2+\cdots+u_n)\end{aligned}$$

with equality if and only if $u_1=u_2=\cdots=u_n$. Since $\langle c, u\rangle \geqq 0$, we obtain $\langle m, u\rangle > 0$ if the components of u are not all equal. While if the components of u are all equal and not all zero, then since $u \in C^*$, the components of u are all positive, and again $\langle m, u\rangle > 0$. This proves that m is an interior point of C, and completes the proof of "if".

For the "only if" part, suppose that, for $U=\{1, 2, \cdots, r\}$, $N(U)/r \geqq k/n$. We will show m cannot be an interior part of C. Consider the vector u whose first r components are equal to $n-r$, and whose remaining $n-r$ components are equal to $-r$. For any exponent $c=(c_1, c_2, \cdots, c_n)$, we have

$$\begin{aligned}\langle c, u\rangle &= (c_1+c_2+\cdots+c_r)(n-r)-(c_{r+1}+\cdots+c_r)r \\ &= (c_1+c_2+\cdots+c_r)\cdot n - kr.\end{aligned}$$

As c runs through all exponents of P, $\langle c, u\rangle$ will be minimum when $c_1+c_2+\cdots+c_r$ is as small as possible, i.e., when $c_1+c_2+\cdots+c_r=N(U)$. Since $N(U) \geqq kr/n$, we have always $\langle c, u\rangle \geqq 0$ for any exponent c. Hence $u \in C^*$; but $\langle m, u\rangle = 0$ and this proves m is not an interior point of C, and completes the proof.

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK 14850

DEPARTMENT OF MATHEMATICS, THE HEBREW UNIVERSITY, JERUSALEM, ISRAEL