SHORTER NOTES

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A NOTE ON ANALYTIC MEASURES

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Abstract. A corollary to Forelli's generalization of the F. and M. Riesz theorem is proved. It extends Bochner's result concerning the absolute continuity of measures on the torus whose Fourier-Stieltjes coefficients vanish outside a sector of opening less than $\pi$.

In this note, $X$ will be a left coset space $G/H$ where $G$ is a connected Lie group and $H$ is a closed subgroup. Each vector $V$ in the Lie algebra $\mathfrak{g}$ of $G$ determines a one-parameter group $\{T_t\}_{t \in \mathbb{R}}$ of homeomorphisms on $X$ via the formula $T_t(gH) = \exp(tV)gH$, $t \in \mathbb{R}$, $gH \in X$. We shall refer to $\{T_t\}_{t \in \mathbb{R}}$ as the flow determined by $V$. If $\{T_t\}_{t \in \mathbb{R}}$ is a flow on $X$ and if $\mu$ belongs to $M(X)$, the space of finite regular Baire measures on $X$, then $\mu$ is called analytic with respect to $\{T_t\}_{t \in \mathbb{R}}$ in case the function of $t$, $\int_X \phi \circ T_{-t} \, d\mu$, lies in $L^1(C_0(X))$ for each $\phi$ in $C_0(X)$. A measure $\mu$ in $M(X)$ is called quasi-invariant with respect to $\{T_t\}_{t \in \mathbb{R}}$ in case $|\mu|$ and $\mu \circ T_t$ have the same null sets for each $t$ in $\mathbb{R}$ ($|\mu|$ = total variation measure of $\mu$). In [3] Forelli proved that an analytic measure is quasi-invariant. In this note we prove a corollary to Forelli's theorem which extends Bochner's theorem [1, Theorem 5] and a theorem of de Leeuw and Glicksberg [2, Theorem 3.4].

Theorem. Let $V_1, \ldots, V_n$ be a basis for $\mathfrak{g}$ and let $\{T_i^{(t)}\}_{t \in \mathbb{R}}$ be the flow on $X$ determined by $V_i$, $i = 1, \ldots, n$. If $\mu$ is a nonzero measure in $M(X)$ which is analytic with respect to each flow $\{T_i^{(t)}\}_{t \in \mathbb{R}}$, then $\mu$ is equivalent to the transplant to $X$ of (left or right) Haar measure on $G$.

Proof. By a theorem of Mackey [4, Theorem 1.1] it suffices to show that $|\mu|(gE) = 0$ for each null set $E$ for $|\mu|$ and for each $g$ in $G$. For $g$ in $G$,
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Define $T_g$ by the formula

$$T_g(xH) = (gx)H, \quad xH \in X.$$  

First we show that $|\mu| \circ T_g$ moves continuously through $M(X)$. Since the map $\lambda \mapsto \lambda \circ T_g, g \in G, \lambda \in M(X)$ defines an isometric representation of $G$ on $M(X)$, it suffices to check continuity at the identity $e$ of $G$. By hypothesis and Theorem 4 in [3], $|\mu| \circ T_i^{(1)}$ moves continuously through $M(X)$ for each $i$. Hence, as a calculation reveals, the map

$$(t_1, \cdots, t_n) \mapsto |\mu| \circ T_i^{(1)} \circ T_i^{(2)} \cdots \circ T_i^{(n)}$$

is continuous from $\mathbb{R}^n$ into $M(X)$. By the inverse function theorem, there is a neighborhood $N_0$ of the origin in $\mathbb{R}^n$ and a neighborhood $N_e$ of $e$ in $G$ such that the map

$$(t_1, \cdots, t_n) \mapsto \exp(t_1V_1)\exp(t_2V_2) \cdots \exp(t_nV_n)$$

is a diffeomorphism from $N_0$ onto $N_e$. The continuity at $e$ of the map $g \mapsto |\mu| \circ T_g$ follows. Let $E$ be a Baire set in $X$. Since $|\mu| \circ T_g$ moves continuously through $M(X)$, the set of $g$ such that $|\mu| \circ T_g(E) = 0$ is closed. Since $\mu$ is quasi-invariant with respect to each $\{T_i^{(t)}\}_{t \in \mathbb{R}}$, it follows that if $|\mu| \circ T_g(E) = 0$ for some $g$ in $N_e$, then $|\mu| \circ T_g(E) = 0$ for all $g$ in $N_e$. Hence, since $g \mapsto |\mu| \circ T_g$ is continuous, the set of all $g$ such that $|\mu| \circ T_g(E) = 0$ is open. Since $G$ is connected, this set is either all of $G$ or empty, and the proof is complete.

References

1. S. Bochner, Boundary values of analytic functions in several variables and of almost periodic functions, Ann. of Math. (2) 45 (1944), 708–722. MR 6, 124.

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