

ZERO-DIMENSIONAL COMPACT ASSOCIATIVE DISTRIBUTIVE UNIVERSAL ALGEBRAS¹

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ABSTRACT. We consider the question under which conditions a zero-dimensional compact universal algebra \mathfrak{A} of finite type is profinite in the sense that the intersection of all closed congruences on \mathfrak{A} with finite quotients is trivial. This is known ([2], [9]) to be the case for typical algebras such as groups, semigroups, Boolean lattices (or distributive lattices) and associative rings, but not in general. In this paper we show that if the underlying algebra \mathfrak{c} of the \mathfrak{A} has generalized associativity and distributivity (see definitions in §1), then \mathfrak{A} is always profinite. It then follows directly from [2] that the two categories of all residually finite associative, distributive universal algebras of the same finite type and of all zero-dimensional compact ones are in adjoint situation. From this it is shown that all projectives in the latter category are completely characterized in terms of free algebras in the former category.

1. Terminologies and definitions. In this paper, terminologies and definitions mainly follow those in [5] for universal algebra, those in [8] for category theory and those in [2] for the others. Throughout this paper all universal algebras will be of finite type, i.e., by a τ -algebra we mean a pair $(A, (f_i)_{i \in I})$, where A is a set, $(f_i)_{i \in I}$ a family of operations: maps f_i of A^{m_i} into A , m_i the arity of the f_i that is a nonnegative integer and I is a finite ordered set $\{1, 2, \dots, t\}$. And we say that it is of arity type $\tau = (m_1, m_2, \dots, m_t)$.

By a topological τ -algebra we mean a triple $(A, (f_i)_{i \in I}, \mathcal{T})$, where $(A, (f_i)_{i \in I})$ is a τ -algebra and \mathcal{T} is a Hausdorff topology on A under which each f_i is a continuous map of the product space $(A^{m_i}, \mathcal{T}^{m_i})$ into (A, \mathcal{T}) .

Let \mathfrak{A} be a τ -algebra $(A, (f_i)_{i \in N}, (u_p)_{p \in M}, (n_r)_{r \in L})$, where $(f_i)_{i \in N}$ is the family of all operations with the arity $m_i \geq 2$, $(u_p)_{p \in M}$ the family of all

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unary operations ($m_p=1$) and $(n_r)_{r \in L}$ all nullary ones ($m_r=0$), and $N = \{1, 2, \dots, n\}$, $M = \{n+1, \dots, m\}$ and $L = \{m+1, \dots, t\}$. \mathfrak{A} is said to be associative if for each f_i ($i \in N$)

$$\begin{aligned} & f_i(f_i(a_1, a_2, \dots, a_{m_i}), a_{m_i+1}, \dots, a_{2m_i-1}) \\ &= f_i(a_1, f_i(a_2, \dots, a_{m_i+1}), a_{m_i+2}, \dots, a_{2m_i-1}) \dots \\ &= f_i(a_1, \dots, f_i(a_{m_i}, \dots, a_{2m_i-1})) \end{aligned}$$

for all $a_k \in A$, $k=1, 2, \dots, 2m_i-1$, and for each u_p , there exist two integers $n(p) \geq 0$ and $k(p) \geq 1$ such that

$$u_p^{n(p)+k(p)}(a) = u_p^{n(p)}(a)$$

for all $a \in A$.

\mathfrak{A} is said to be distributive if the following (i), (ii) and (iii) (or (i), (ii)' and (iii)) are satisfied:

(i) For each pair of f_i and f_j with $i < j$ in N ,

$$\begin{aligned} & f_j(a_1, \dots, a_{k-1}, f_i(b_1, \dots, b_{m_i}), a_{k+1}, \dots, a_{m_j}) \\ &= f_i(f_j(a_1, \dots, a_{k-1}, b_1, a_{k+1}, \dots, a_{m_j}), \dots, \\ & \qquad \qquad \qquad f_j(a_1, \dots, a_{k-1}, b_{m_i}, a_{k+1}, \dots, a_m)) \end{aligned}$$

for all a_m and $b_n \in A$ and all k ($k=1, 2, \dots, m_j$).

(ii) For each f_i and each u_p , either

$$u_p(f_i(a_1, \dots, a_{m_i})) = f_i(u_p(a_{m_i}), \dots, u_p(a_1))$$

or

$$u_p(f_i(a_1, \dots, a_{m_i})) = f_i(a_1, \dots, a_{k-1}, u_p(a_k), a_{k+1}, \dots, a_{m_i})$$

for some k ($k=1, 2, \dots, m_i$).

(ii)' For each f_i and each u_p

$$u_p(f_i(a_1, \dots, a_{m_i})) = f_j(u_p(a_1), \dots, u_p(a_{m_i}))$$

for some f_j with $m_i = m_j$.

(iii) For each pair of u_p and u_q , $u_q(u_p(a)) = u_p(u_q(a))$ for all $a \in A$.

Obvious classes of this type of algebras are, for example, sets with trivial operations, semigroups, distributive lattices, groups, rings, and lattice-ordered (semi) groups. Boolean lattices are of type (i), (ii)', and (iii). For a ring, f_1 is the addition, f_2 the multiplication, u_3 the unary operation of the inverse of f_1 and N is the ordered set $\{1, 2\}$.

Let $\mathfrak{A} = (A, (f_i)_{i \in I})$ be a τ -algebra. By a residually finite congruence relation θ we mean a congruence θ such that the cardinal $|\mathfrak{A}/\theta|$ of the quotient algebra of \mathfrak{A} by θ is finite. An algebra \mathfrak{A} is said to be residually finite [1] if the intersection of all residually finite congruences of \mathfrak{A} is

trivial, namely, the diagonal Δ_A of A . For example, every set with trivial operations, every distributive lattice and every Boolean lattice are always residually finite.

2. Zero-dimensional compact algebras. Let \mathcal{A} be an equational class of associative and distributive τ -algebras. Let $K\mathcal{A}$ ($K_0\mathcal{A}$) be the class of all topological τ -algebras whose underlying spaces are compact (respectively, compact zero-dimensional) and whose underlying τ -algebras belong to \mathcal{A} , and $F\mathcal{A}$ the class of all finite τ -algebras belonging to \mathcal{A} . If each algebra in $F\mathcal{A}$ is given the discrete topology of its underlying set, then we have $F\mathcal{A} \subset K_0\mathcal{A} \subset K\mathcal{A}$. A closed congruence θ on $\mathfrak{U} \in K\mathcal{A}$ is said to have finite quotient if the quotient \mathfrak{U}/θ belongs to $F\mathcal{A}$. If the intersection of all closed congruences on \mathfrak{U} with finite quotient is trivial then \mathfrak{U} is called a profinite τ -algebra. Let $\text{Pro } F\mathcal{A}$ be the class of all profinite τ -algebras in $K\mathcal{A}$. It is known that $\text{Pro } F\mathcal{A} \subset K_0\mathcal{A}$. Regarding the identity $\text{Pro } F\mathcal{A} = K_0\mathcal{A}$, we know that groups, semigroups, distributive lattices and associative rings are always satisfied; on the other hand, there is the following counterexample [2]: the Stone-Čech compactification βN of the set of all integers N together with the continuous extension of the successor function: $n \rightarrow n + 1$ is a zero-dimensional compact algebra, of type (1), but not profinite since it has only countably many open-closed congruences and βN is not metrizable. Now we show in the following theorem that $\text{Pro } F\mathcal{A} = K_0\mathcal{A}$ for suitable \mathcal{A} .

THEOREM 1. *If \mathfrak{U} is a zero-dimensional compact associative distributive τ -algebra, where $\tau = (m_1, \dots, m_k)$, then \mathfrak{U} is profinite.*

Before the proof of the theorem, we give several lemmas and notation. The following lemma is well known.

LEMMA 1. *If X is a zero-dimensional compact topological space and G is an open subset in the product space $X \times X$ containing the diagonal Δ_X then there exists a closed and open equivalence relation R contained in G .*

To simplify we introduce the following notation:

For $\mathfrak{U} = (A, f_i)_{i \in N}, (u_p)_{p \in M}, (n_r)_{r \in L} \in K_0\mathcal{A}$ let R be an equivalence relation on A .

Let (a, b) be a pair of elements in A .

For an operation f_i ($i \in N$)

$$(1) \quad f_i[a] R f_i[b] \quad \text{iff} \quad f_i(x_1, \dots, x_{k-1}, a, x_{k+1}, \dots, x_{m_i}) \\ \equiv f_i(x_1, \dots, x_{k-1}, b, x_{k+1}, \dots, x_{m_i})(R)$$

for all $x_l \in A$ ($l=1, 2, \dots, k-1, k+1, \dots, m_i$) and all k ($k=1, 2, \dots, m_i$).

$$(2) \quad f_i f_j [a] R f_i f_j [b] \text{ iff } f_i [f_j [a]] R f_i [f_j [b]],$$

namely,

$$\begin{aligned} & f_i(x_1, \dots, x_{k-1}, f_j(y_1, \dots, y_{l-1}, a, y_{l+1}, \dots, y_{m_j}), x_{k+1}, \dots, x_{m_i}) \\ & \equiv f_i(x_1, \dots, x_{k-1}, f_j(y_1, \dots, b, \dots, y_{m_j}), x_{k+1}, \dots, x_{m_i})(R) \end{aligned}$$

for all x_m and $y_h \in A$, and all k and l ($k=1, 2, \dots, m_i, l=1, 2, \dots, m_j$) \dots .

$$(r) \quad \begin{aligned} & f_i f_j \dots f_p [a] R f_i f_j \dots f_p [b] \\ & \text{iff } f_i [f_j \dots [f_p [a]] \dots] R f_i [f_j \dots [f_p [a]] \dots]. \end{aligned}$$

LEMMA 2. *If R is an equivalence relation on $\mathfrak{A} = (A, (f_i)_{i \in N}, (u_p)_{p \in M}, (n_r)_{r \in L})$ then $f_i f_j [a] R f_i f_j [b]$ implies $f_i f_j f_i [a] R f_i f_j f_i [b]$ for each pair (a, b) of elements in A .*

PROOF. Let (a, b) satisfy $f_i f_j [a] R f_i f_j [b]$.

By the associative law, we have that, for each $j, k, l \in \{1, 2, \dots, m_i\}$,

$$\begin{aligned} & f_i(x_1, \dots, x_j, f_i(y_1, \dots, y_k, f_i(z_1, \dots, z_l, a, \dots, z_{m_i}) \dots y_{m_i}) \dots x_{m_i}) \\ & = f_i(f_i(x_1, \dots, x_j, y_1, \dots, y_k, z_1, \dots, z_l, a, \dots), \dots, f_i(z_{m_i}, \dots), \dots, x_{m_i}) \\ & \hspace{15em} \text{if } j + k + l + 1 \leq m_i, \\ & = f_i(f_i(f_i(x_1, \dots, x_j, y_1, \dots, y_k, z_1, \dots, z_l, a, \dots, z_{m_i}, \dots, y_{m_i}, \dots, x_{m_i}) \\ & \hspace{15em} \text{if } j + k + l + 1 > m_i. \end{aligned}$$

In either case, by hypothesis, the expression is R -equivalent to the corresponding expression replacing a by b . Hence we have $f_i f_j f_i [a] R f_i f_j f_i [b]$.

LEMMA 3. *With the same notation as in Lemma 2,*

$$\begin{aligned} & f_j \dots f_k f_i f_l \dots f_m [a] R f_j \dots f_k f_i f_l \dots f_m [b] \\ \text{implies} & \\ & f_j \dots f_k f_i f_i f_l \dots f_m [a] R f_j \dots f_k f_i f_i f_l \dots f_m [b]. \end{aligned}$$

PROOF. Let a' be an arbitrary expression in $f_i \dots f_m [a]$ and b' its corresponding expression replacing a by b in $f_i \dots f_m [b]$. By the same argument as in the proof of Lemma 2, it follows that $f_j \dots f_k f_i f_i [a'] R f_j \dots f_k f_i f_i [b']$. Hence we have the required result.

PROOF OF THEOREM 1. Let $\mathfrak{A} = (A, (f_i)_{i \in N}, (u_p)_{p \in M}, (n_r)_{r \in L}, \mathcal{T})$, where $N = \{1, 2, \dots, n\}$ and $M = \{n+1, \dots, m\}$. By Lemma 1, \mathfrak{A} has a closed and open equivalence relation R such that $R \subset G$ for a given open subset G in $(A \times A, \mathcal{T} \times \mathcal{T})$ containing Δ_A .

Let η be the set of all pairs $(a, b) \in R$ satisfying the following $2n$ conditions $(C_1) \sim (C_{2n})$:

$$(C_1): f_i[a] R f_i[b] \text{ for } i=1, 2, \dots, n,$$

$$(C_2): f_i f_j[a] R f_i f_j[b] \text{ for all } i \text{ and } j \in N \text{ with } i \leq j, \dots$$

$$(C_r): f_i f_j \dots f_l[a] R f_i f_j \dots f_l[b] \text{ for all } i, j, \dots, l \in N \text{ with } i \leq j \leq \dots \leq l, \text{ where the number of operations } f_i, \dots, f_l \text{ is } r, \dots$$

$$(C_{2n}): f_1 f_1 f_2 f_2 \dots f_n f_n[a] R f_1 f_1 f_2 f_2 \dots f_n f_n[b],$$

where all indices $i, j, \dots, l \in N$ in $(C_1) \sim (C_{2n})$ are allowed to repeat not more than twice. Then η is a congruence on the algebra $(A, (f_i)_{i \in N})$ without unary and nullary operations. In fact, since η is clearly an equivalence relation on A , it suffices to show that for each pair $(a, b) \in \eta$, $f_k[a] \eta f_k[b]$ is satisfied for all $k \in N$ [5]. To this end, we show that the pair $(f_k[a], f_k[b])$ satisfies $(C_1) \sim (C_{2n})$. Firstly, $f_k[a] R f_k[b]$ is true by (C_1) . To check the condition (C_1) of the pair $(f_k[a], f_k[b])$, we observe two cases:

$$(1) \text{ if } i \leq k \text{ in } N, \text{ then by } (C_2) f_i f_k[a] R f_i f_k[b],$$

(2) if $k < i$ in N then by the distributivity of f_k and f_i , we may interchange f_i and f_k so that we can apply (C_2) to obtain $f_i f_k[a] R f_i f_k[b]$.

Now to check the condition (C_r) , we have to consider several cases:

(1) if $i \leq \dots \leq l \leq k$ and k is not repeated more than twice, then by (C_{r+1}) we have directly $f_i f_j \dots f_l f_k[a] R f_i f_j \dots f_l f_k[b]$, (2) if $i \leq \dots \leq l \leq k$ and k is repeated three times, then by Lemma 3 and (C_r) we have the same result, (3) if $l \not\leq k$, then by the distributivity and Lemma 3, we may arrange the indices $\{i, \dots, l, k\}$ in the natural order and k is not repeated more than twice so that one can apply (C_r) or (C_{r+1}) to obtain the same result. Finally, for (C_{2n}) , k must be repeated three times. Therefore applying the distributivity first and Lemma 3 second we have $f_1 f_1 \dots f_n f_n f_k[a] R f_1 f_1 \dots f_n f_n f_k[b]$ by (C_{2n}) . Hence η is a congruence on $(A, (f_i)_{i \in N})$.

Now we show further that η is closed and open in $(A \times A, \mathcal{F} \times \mathcal{F})$. Let $(a, b) \in \eta$. Since R is open and all f_i are continuous, there exist open neighborhoods U_a and V_b of a and b , respectively, such that $U_a \times V_b \subset \eta$. For each given expression in (C_r) we can find open neighborhoods U and V of a and b , respectively, such that every pair of elements in $U \times V$ satisfies the given expression. Since the number of all different expressions in (C_r) ($r=1, 2, \dots, 2n$), is at most finite, the intersections U_a and V_b of all such neighborhoods U of a and V of b , respectively, are still open, and $U_a \times V_b \subset \eta$. Thus η is open. It is easy to see that η is closed since R is.

Now we construct a closed and open congruence θ on the same algebra \mathfrak{A} such that $\theta \subset \eta$. Let θ be the set of all pairs (a, b) in η satisfying the following conditions:

$$(D_1) \sim (D_{s-1}):$$

$$(D_r): u_p(u_q(\dots(u_t(a))\dots)) \equiv u_p(u_q(\dots(u_t(b))\dots))(\eta), r=1, 2, \dots, s-1,$$

where $s = (n(p) + k(p)) + \dots + (n(t) + k(t))$ and each index p in (D_r) , $r = 1, 2, \dots, s - 1$, may be repeated a number of times less than $n(p) + k(p)$. Then by the distributivities (ii) and (iii) (or (ii)' and (iii)'), it is not difficult to see that θ is also a closed and open congruence on \mathfrak{A} , and $\theta \subset \eta \subset R \subset G$. Furthermore, θ is residually finite since θ is open and \mathcal{F} is compact. On the other hand, the set G is an arbitrary open subset in $(A \times A, \mathcal{F} \times \mathcal{F})$ containing Δ_A , and \mathcal{F} is Hausdorff. Thus the intersection of all closed and open congruences on \mathfrak{A} is trivial. It follows that the $\mathfrak{A} = (A, (f_i)_{i \in N}, (u_p)_{p \in M}, (n_r)_{r \in L}, \mathcal{F})$ is profinite.

REMARK. The hypotheses of associativity and distributivity in Theorem 1 are not necessary, for example, profinite nondistributive lattice etc. However, it is clear by Birkhoff's Subdirect Representation Theorem that for an equational class \mathcal{A} if each subdirectly irreducible τ -algebra in $F\mathcal{A}$ is associative and distributive, then $\text{Pro } F\mathcal{A} = K_0\mathcal{A}$ iff all \mathfrak{A} in $K_0\mathcal{A}$ are associative and distributive.

3. **Projectives in $K_0\mathcal{A}$.** Let \mathcal{A} be an equational class of associative, distributive τ -algebras, and let \mathcal{A}_0 be the class of all residually finite τ -algebras in \mathcal{A} . Then \mathcal{A}_0 is evidently hereditary and productive in \mathcal{A} . Consequently, it has free algebras over \mathcal{A}_0 with any set as the free generating set. For $\mathfrak{A} \in \mathcal{A}_0$, the family of all congruences θ with $\mathfrak{A}/\theta \in F\mathcal{A}$ forms a basis of a separated uniformity (called the $F\mathcal{A}$ -adic uniformity) on the underlying set A of \mathfrak{A} [2]. The completion of the resulting uniform space is called the $F\mathcal{A}$ -adic completion of A , denoted by PA . Since $\text{Pro } F\mathcal{A} = K_0\mathcal{A}$ by Theorem 1, we have the following lemma from [2]:

LEMMA 4. *The functor P of $F\mathcal{A}$ -adic completion is left adjoint to the underlying algebra functor $U: K_0\mathcal{A} \rightarrow \mathcal{A}_0$.*

By a projective \mathfrak{B} in K_0 , we mean that for an onto morphism $f: \mathfrak{A} \rightarrow \mathfrak{B}$ and a morphism $g: \mathfrak{B} \rightarrow \mathfrak{B}$ in $K_0\mathcal{A}$, there exists $h: \mathfrak{B} \rightarrow \mathfrak{A}$ in $K_0\mathcal{A}$ such that $fh = g$.

THEOREM 2. *\mathfrak{B} is projective in $K_0\mathcal{A}$ iff \mathfrak{B} is a retract of the $F\mathcal{A}$ -adic completion of a free algebra over \mathcal{A}_0 .*

PROOF. Suppose that \mathfrak{B} is projective in $K_0\mathcal{A}$. Let \mathcal{F} be the free algebra over \mathcal{A}_0 generated by the underlying set S of \mathfrak{B} . Let $U\mathfrak{B}$ be the underlying algebra of \mathfrak{B} . Then we have an onto homomorphism $h: \mathcal{F} \rightarrow U\mathfrak{B}$ such that $h|_S = 1_S$. The map h , being uniformly continuous with respect to the $F\mathcal{A}$ -adic uniformities, extends uniquely to an onto morphism $Ph: P\mathcal{F} \rightarrow P(U\mathfrak{B})$ in $K_0\mathcal{A}$ [2]. Moreover, there exists an onto morphism $g: P(U\mathfrak{B}) \rightarrow \mathfrak{B}$, namely the back adjunction for \mathfrak{B} . Since \mathfrak{B} is projective,

for the composition $g(Ph)$ and the identity $1_{\mathfrak{B}}$ of \mathfrak{B} , we have a morphism $\varphi: \mathfrak{B} \rightarrow P\mathcal{F}$ such that $g(Ph)\varphi = 1_{\mathfrak{B}}$. Thus \mathfrak{B} is a retract of $P\mathcal{F}$. The converse is directly from Lemma 4 and the fact that left adjoint functor preserves projectivity provided that the right adjoint preserves the onto morphisms [7].

REMARK. For any equational class \mathcal{A} , Lemma 4 is still true if $K_0\mathcal{A}$ is replaced by $\text{Pro } F\mathcal{A}$ [2]. Thus Theorem 2 is, after all, true for any \mathcal{A} with $\text{Pro } F\mathcal{A}$ in the place of $K_0\mathcal{A}$.

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