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PROFINITE GROUPS ARE GALOIS GROUPS

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ABSTRACT. Artin's theorem on finite automorphism groups of fields extends to profinite groups, and hence every profinite group is a galois group.

It is well known that every finite group is the galois group of some field extension, but the corresponding statement about profinite groups does not seem to be on record. It is proved here by generalizing Artin's theorem that finite automorphism groups are galois groups.

THEOREM 1. *Let F be a field and G a profinite group. Assume that*
(i) *G acts as a group of automorphisms of F,*
(ii) *the action is faithful, and*
(iii) *the stabilizer of each element of F is an open subgroup.*
Let K be the field of fixed elements. Then F is a normal separable algebraic extension of K, and G is the galois group of the extension.

PROOF. Let $x_1, \cdots, x_n$ be any finite set of elements in F. Let H_i be the open subgroup fixing x_i, so that the orbit Gx_i corresponds to the H_i-cosets. Let N be the intersection of all conjugates of the various H_i, a finite intersection which is then also open. The subfield $L=K(Gx_1, \cdots, Gx_n)$ is mapped to itself by G, and N is the subgroup acting trivially on L; thus the finite group G/N acts faithfully on L, and the field of fixed elements is still K. By the usual Artin theorem [**1**, p. 194], L is a finite normal separable extension of K, and G/N is its galois group.

Since F is a directed union of such fields, it is a normal separable algebraic extension of K. Thus $\mathrm{Gal}(F/K)$ is defined, and by assumption G maps continuously and injectively into it. Since G is compact, the image is closed; it is also dense, since G maps onto all the groups $\mathrm{Gal}(L/K)$. Thus $G \to \mathrm{Gal}(F/K)$ is an isomorphism.

THEOREM 2. *Let G be a profinite group. Then it is the galois group of some field extension.*

PROOF. Let X be the disjoint union of the sets G/H for all open subgroups H. Then G acts faithfully on X, and each element of X has open

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stabilizer. Let k be any field; take the elements of X as indeterminates, and form the pure transcendental extension $F=k(X)$. Then G acts on F as in Theorem 1.

The characteristic of k, and hence of F, can of course be arbitrary.

Reference

1. S. Lang, *Algebra*, Addison-Wesley, Reading, Mass., 1965. MR 33 #5416.

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