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AN EXTENSION THEOREM FOR H^p FUNCTIONS

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ABSTRACT. Let V be a pure $(n-1)$-dimensional variety in the polydisc U^n with the distance from V to the torus Π^n positive and assume f is analytic on $\Omega \equiv U^n \setminus V$. Further let $u(z)$ be the real part of a function g analytic on Ω and assume $|f(z)|^p \leqq u(z)$ for $z \in \Omega$. Then f can be analytically extended to a function $\hat{f}$ in $H^p(U^n)$.

1. **Notation and definitions.** In this note, C will denote the complex field and C^n will be the cartesian product of n copies of C. The open unit disc in C is denoted by U; its boundary is the circle T. The unit polydisc U^n and the torus T^n are the subsets of C^n which are cartesian products of n copies of U and T, respectively. If $Z=(z_1, \cdots, z_n) \in C^n$ the norm of Z is given by

$$\|Z\| = \sup_{1 \leq i \leq n} |z_i|$$

and the polydisc of radius $\rho > 0$ with center $Z^0 = (z_1^0, \cdots, z_1^n)$ is given by

$$U_\rho^n(Z^0) = \{Z \in C^n : \|Z - Z^0\| < p\}.$$

A complex-valued function f defined on U^n is said to be holomorphic in U^n provided f is continuous in U^n and f is holomorphic in each variable separately. For $1 \leqq p < \infty$ the Banach space $H^p(U^n)$ consists of those holomorphic functions f which satisfy

$$\sup_{0<r_i<1} \int_{T^n} |f(r_1 e^{ix_1}, \cdots, r_n e^{ix_n})|^p \, dm_n(x) < \infty, \tag{1}$$

where m_n is the normalized Haar measure on T^n. A variety in U^n is a closed subset $V \subset U^n$ such that for each point $p \in V$ there is a neighborhood W of p, $W \subseteq U^n$ and analytic functions $f_1, \cdots, f_k$ (analytic in W) such that

$$V \cap W = Z(f_1) \cap \cdots \cap Z(f_k)$$

where $Z(f)$ is the zero set of f. If each of the irreducible branches of V

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have dimension $n-1$ then V is a pure $(n-1)$-dimensional variety. A theorem of Cartan (see [**1**, p. 251]) implies that if V is a pure $(n-1)$-dimensional variety in U^n there is a function g analytic in U^n with $V=Z(g)$. The set $RP(U^n)$ denotes the set of real-valued functions defined on U^n which are the real parts of functions holomorphic on U^n. A real-valued function $a(Z)$ defined on U^n is n-harmonic if for each fixed $Z=(z_1, \cdots, z_n)$ in U^n the mapping

$$\zeta_j \to a(z_1, z_2, \cdots, \zeta_j, \cdots, z_n)$$

is harmonic in U for each $j=1, 2, \cdots, n$. Each function in $RP(U^n)$ is n-harmonic in U^n but there are n-harmonic functions which are not the real parts of functions holomorphic in U^n.

2. **An extension theorem.** Let Ω be a domain in the complex plane $\boldsymbol{C}$. For $1 \leqq p < \infty$, let $H^p(\Omega)$ be the space of functions f, holomorphic on Ω, and such that the subharmonic function $|f(Z)|^p$ has a harmonic majorant in D (see [**4**, p. 46]). If Z_0 is a fixed point of Ω and $u(Z)$ is the least harmonic majorant of $|f(Z)|^p$, then $\|f\| = [u(Z_0)]^{1/p}$ can be taken as the norm for $f \in H^p(\Omega)$. It is easily seen that with $n=1$ and $\Omega = U$ that this gives the same set of functions as those satisfying equation (1).

The following lemma appears to be known but since we have no specific reference we outline a short proof.

LEMMA 1. *Assume $f(z)$ is holomorphic in $0<|z|<1$ and that there exists a function $u(z)$ harmonic on $0<|z|<1$ such that f satisfies $|f(z)| \leqq u(z)$ there. Then f can be extended to be a holomorphic function on U.*

PROOF. Assume first that f has a pole at $z=0$, say $f(z)=Z^{-k}S(z)$ with $S(0) \neq 0$ and k a positive integer. It is possible to select a (multivalued) function $V(z)$ so that $g(z) \equiv \exp(u(z)+iV(z))$ is analytic in $0<|z|<1$. In this case g will have a pole at $z=0$ and also g has growth rate $\exp(r^{-k})$, which is not possible.

If f has an essential singularity at $z=0$, we conclude from the inequalities

$$|e^f| \leqq e^{|f|} \leqq |g| = e^u \quad \text{and} \quad e^{|f|} \geqq 1$$

that g has a pole at $z=0$. This implies the existence of an integer q such that the function $z^q e^f$ is bounded near $z=0$. This is not possible.

In his thesis [**2**], Parreau has proven the following result. If K is a compact subset of U with the logarithmic capacity of K being zero, then each $f \in H^p(U \backslash K)$ can be continued to a function $\hat{f}$ analytic on U and $\hat{f}$ is in $H^p(U)$. The following theorem is our analogue of Parreau's theorem for U^n.

THEOREM 1. *Let V be a pure $(n-1)$-dimensional variety in U^n with the distance of V to T^n being positive. If f is holomorphic in $\Omega \equiv U^n \backslash V$ and if there is a function $u(Z) \in RP(\Omega)$ which satisfies the inequality $|f(Z)|^p \leqq u(Z)$ for all $Z \in \Omega$ then there is a function $\hat{f}$ which extends f holomorphically to all of U^n and such that $\hat{f}$ is in $H^p(U^n)$.*

PROOF. Without loss of generality we assume V is the zero set of a function g analytic on U^n. Let us prove first that f can be extended analytically to points $Z^0=(z_1^0, \cdots, z_n^0)$ of $V=Z(g)$. We assume $Z^0 \in V$. There is a nonsingular linear change of coordinates in C^n which will make g regular of order k in z_n at the point Z^0. That is, there is a positive number p such that

$$g(Z) = \Lambda(Z)Q(z_1, \cdots, z_n),$$

where $\Lambda(Z)$ is holomorphic and nonzero in the polydisc $U_p^n(Z_0)$ and Q is a polynomial of degree k in the variable $z_n - z_n^0$. The coefficients of Q are holomorphic functions $A_j = A_j(z_1, \cdots, z_{n-1})$ of $z_1, \cdots, z_{n-1}$ which vanish at $(z_1^0, \cdots, z_{n-1}^0)$. From this representation one can find two positive numbers p_1 and p_2 ($p_i \leqq p$) such that if

$$\zeta = (\zeta_1, \cdots, \zeta_{n-1}) \in U_{p_1}^{n-1}(z_1^0, \cdots, z_{n-1}^0)$$

then there are precisely k points $\zeta_n^1(\zeta), \zeta_n^2(\zeta), \cdots, \zeta_n^k(\zeta)$ which satisfy

$$|\zeta_n^i(\zeta) - z_n^0| < p_2 \quad \text{and} \quad g(\zeta, \zeta_n^i(\zeta)) = 0.$$

The numbers p_1 and p_2 can also be chosen so that $g(\zeta, \zeta_n) \neq 0$ if

$$\zeta \in U_{p_1}^{n-1}(z_1^0, \cdots, z_{n-1}^0) \quad \text{and} \quad |\zeta_n - z_n^0| = p_2.$$

If $\zeta \in U_{p_1}^{n-1}(z_1^0, \cdots, z_{n-1}^0)$ is fixed the set $\{\zeta_n^j(\zeta)\}_{j=1,2,\cdots,k}$ is finite and hence we may apply Lemma 1 to conclude that $f(\zeta, \zeta_n)$ is holomorphic in ζ_n, $|\zeta_n - z_n^0| \leqq p_2$. Define the function

$$F(\zeta, \zeta_n) = \frac{1}{2\pi i} \int_{z_n - z_n^0 = p_2} \frac{f(\zeta, z_n)\, dz_n}{(z_n - \zeta_n)}$$

on $\zeta \in U_{p_1}^{n-1}(z_1^0, \cdots, z_{n-1}^0)$ and $|\zeta_n - z_n^0| < p_2$. It is holomorphic on its domain of definition. By the Cauchy integral theorem $F(\zeta, \zeta_n) = f(\zeta, \zeta_n)$, so that we have extended f to a holomorphic function in a neighborhood of $Z^0 \in V$.

We show next that the extended function f is in $H^p(U^n)$. Our hypothesis implies the existence of an annulus

$$A(r) = \prod_{i=1}^{n} \{r_i \leqq |z_i| < 1\}$$

which is contained in Ω. We have also assumed the existence of a function $h(Z)$ which is holomorphic on Ω such that $\operatorname{Re} h(Z)=u(Z)$. The function $h(Z)$ has a Laurent expansion about the origin given by

$$h(Z) = \sum C(k_1, \cdots, k_n) z_1^{k_1} \cdots z_n^{k_n},$$

where $Z \in A(r)$, the exponents $k_i \in Z$ and the series converges absolutely and uniformly on compacta in $A(r)$. But for $\rho=(\rho_1, \cdots, \rho_n)$ sufficiently close $(1, \cdots, 1)$ we have

$$\int_{T^n} |f(\rho_1 e^{ix_1}, \cdots, \rho_n e^{ix_n})|^p \, dm_n(x) \\ \leqq \operatorname{Re}\left[\sum C(k_1, \cdots, k_n)\left(\int_T \rho_1^{k_1} e^{ik_1x_1} \frac{dx_1}{2\pi}\right) \cdots \left(\int_T \rho_n^{k_n} e^{ik_nx_n} \frac{dx_n}{2\pi}\right)\right].$$

The right side of the above inequality has only a finite number of terms and these are uniformly bounded as $\rho_j \to 1$. The integral means are increasing on ρ and hence they are bounded. This completes the proof.

Remarks. In proving that f has an analytic extension to U^n we did not need that $u(Z) \in RP(\Omega)$ or that the distance from V to T^n was positive. One can obtain the extension for any pure $(n-1)$-dimensional variety B if only $|f(Z)|^p$ has an n-harmonic majorant $u(Z)$ (in Ω). It is known [**3**, p. 52] that f is in $H^p(U^n)$ if and only if $|f|^p$ has an n-harmonic majorant on U^n.

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