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INVARIANT TRACES ON ALGEBRAS

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ABSTRACT. Certain properties of traces on a finite-dimensional associative algebra A lead to the definition of an element $t(A) \in H^1(\mathrm{Out}\, A, C^*)$, C^* being the multiplicative group of the center of A as Out A-module. It is shown that $t(A)=0$ is equivalent to the existence of nondegenerate traces on A which are invariant under composition with all automorphisms of A. In particular, by means of Galois theory, $t(A)=0$ is shown for a semisimple algebra A, whereas $t(A)\neq 0$ for certain group algebras.

1. Let R be a field, A an associative unitary algebra of finite dimension over R. By a *trace* on A we mean a linear map $\tau: A \to R$ such that $\tau(ab)=\tau(ba)$ $\forall a, b \in A$. This is one possible generalization of the notion of a trace on matrix rings (see [4]; for a generalization in another context, see [2]).

In §§2–4 we shall list some generalities on traces; let $T(A)$ be the R-vectorspace of all traces on A.

2. The existence of nonzero traces on A depends on the abelianized algebra A^a. Let $[A, A]$ be the vectorspace generated by all commutators $[a, b]=ab-ba$ in A, A^a the quotient $A/[A, A]$. The class map $\pi: A \to A^a$ provides an isomorphism of vectorspaces

$$\pi^*: \mathrm{Hom}_R(A^a, R) \to T(A),$$

where π^* is the dual map of π.

One knows that $A^a \neq (0)$ if A is simple [1], hence

(2.1) $$T(A) \neq (0) \quad \text{for a simple algebra } A.$$

3. The *radical* of a trace τ is the set

$$R_\tau = \{a \in A / \tau(ab) = 0 \ \forall b \in A\},$$

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and τ is *nondegenerate* if $R_\tau=(0)$. As R_τ is a 2-sided ideal,

(3.1) *a nonzero trace on a simple algebra is nondegenerate.*

$T(A)$ is a module over the center C of A, as $z \cdot \tau$ for $z \in C$ and $\tau \in T(A)$ defined by

$$(z \cdot \tau)(a) := \tau(za), \qquad a \in A, \tag{3.2}$$

is again a trace.

PROPOSITION 1. *A nondegenerate trace $\eta \in T(A)$ is a free generator of the C-module $T(A)$.*

PROOF. η provides a linear isomorphism from A to its dual $\mathrm{Hom}_R(A, R)$; for every $\tau \in T(A) \subset \mathrm{Hom}_R(A, R)$ there exists a unique $b \in A$ such that $\tau(a)=\eta(ba)$ $\forall a \in A$. We then have the following sequence of implications

$$\begin{aligned} \tau(a_1a_2) = \tau(a_2a_1) \qquad & \forall a_i \in A \\ &\Rightarrow \eta(ba_1a_2) = \eta(ba_2a_1) = \eta(a_1ba_2) \qquad \forall a_i \in A \\ &\Rightarrow ba_1 - a_1b \in R\eta = (0) \qquad \forall a_1 \in A \\ &\Rightarrow b \in C \Rightarrow \tau = b \cdot \eta. \end{aligned}$$

COROLLARY 1. *Suppose the set $B(A)$ of nondegenerate traces on A is nonempty. Then, the C-module structure of $T(A)$ defines a simply transitive action of C^* on $B(A)$, where C^* is the multiplicative group of invertible elements of the center C of A.*

4. Let Aut A, In A denote the group of all automorphisms and anti-automorphisms, of all inner automorphisms resp. of A, and denote the quotient group Aut A/In A by Out A. As can be seen immediately from the definitions, composing an (anti-) automorphism with a trace yields again a trace and thus an operation of Aut A on $T(A)$. Inner automorphisms act in this way as the identity, and we finally get an action of Out A on $T(A)$. Let $\tau \cdot \omega$ ($\tau \in T(A)$, $\omega \in \mathrm{Out}\, A$) be the symbol for this action. Its relationship with the C-module structure of $T(A)$ may be described in the form of an associative law

$$(\omega c) \cdot (\tau \cdot \omega^{-1}) = (c \cdot \tau) \cdot \omega^{-1}, \qquad c \in C, \tau \in T(A), \omega \in \mathrm{Out}\, A. \tag{4.1}$$

5. We say that a trace τ is *invariant* if $\tau \cdot \omega = \tau$ $\forall \omega \in \mathrm{Out}\, A$. We are coming now to the main point of this note which consists in giving a condition on the cohomology level for the existence of nondegenerate invariant traces.

In the subsequent statement, C^* is meant to be an Out A-module via the operation of automorphisms on the center.

PROPOSITION 2. *For every algebra A with $B(A)\neq\varnothing$, there is defined an element $t(A)\in H^1(\mathrm{Out}\,A, C^*)$ such that $t(A)=0$ precisely if A has nondegenerate invariant traces.*

PROOF. By Corollary 1, there belongs to every $\tau\in B(A)$ a map $f_\tau:\mathrm{Out}\,A\to C^*$ such that

$$\tau\cdot\omega^{-1}=f_\tau(\omega)\cdot\tau \qquad \forall\omega\in\mathrm{Out}\,A. \tag{5.1}$$

Then, the following statements are immediate consequences of (4.1):

(1) f_τ is a crossed homomorphism.

(2) For τ and $\eta\in B(A)$, f_τ and f_η differ by a principal crossed homomorphism.

If $t(A)$ is then defined as the cohomology class of the f_τ's the statement in Proposition 2 on $t(A)$ is easily verified using again (4.1).

6. **Example 1.** $t(A)=0$ *for a semisimple algebra A*. In fact, if A is simple we know from (2.1) and (3.1) that $B(A)\neq\varnothing$. As $\mathrm{Out}\,A$ is a finite group and C^* the multiplicative group of a field, a fundamental theorem of Galois theory asserts that $H^1(\mathrm{Out}\,A, C^*)=0$ [**3**, Chapter IV, p. 106]. By Proposition 2, A has nondegenerate invariant traces.

If $A=\bigoplus A_i$ $(1\leqq i\leqq n)$ is semisimple with simple components A_i, let

$$\mathrm{Out}_{\,i}A := \{\omega\in\mathrm{Out}\,A/\omega(A_i)\subset A_i\}.$$

Choose one index i for each conjugacy class of the subgroups $\mathrm{Out}_{\,i}A\subset\mathrm{Out}\,A$, and on A_i a nondegenerate invariant trace τ_i. If $\mathrm{Out}_{\,k}A$ is conjugate to $\mathrm{Out}_{\,i}A$, there exists $\alpha\in\mathrm{Aut}\,A$ with $\alpha:A_k\to A_i$, and define τ_k on A_k by $\tau_k=\tau_i\circ\alpha$. The direct sum of all these traces on the different A_i is seen to be a nondegenerate invariant trace on A.

7. **Example 2.** *Let G_p be a finite cyclic group of prime order $p>2$, $R=\boldsymbol{Z}_p$ and A the group algebra $\boldsymbol{Z}_p(G_p)$. Then, $t(A)\neq 0$.*

First we note, that in the more general situation of a finite group G and field R, the group algebra $R(G)$ has at least one nondegenerate trace τ_0 given by $\tau_0(x)=x(1)$ where $x=\sum x(g)\cdot g\in R(G)$, $g\in G$ and $x(g)\in R$, and 1 is the unit in G. Hence, $t(R(G))$ is defined.

Suppose now q is a generator of G_p. As $A=\boldsymbol{Z}_p(G_p)$ is commutative, we have $\mathrm{Out}\,A=\mathrm{Aut}\,A$ and every $\alpha\in\mathrm{Aut}\,A$ is characterized by its value on q. If $x=\alpha(q)$, $x^p=1$ and the powers x^ν, $0\leqq\nu\leqq p-1$, form an R-basis of A. Conversely, every $x\in A$ with this property is the value of some $\alpha\in\mathrm{Aut}\,A$ on q. Therefore at least p automorphisms α_ν, $0\leqq\nu\leqq p-1$, of A exist which are given by their values on q:

$$\alpha_\nu(q)=q^\nu,\qquad 1\leqq\nu\leqq p-1,$$
$$\alpha_0(q)=\tfrac{1}{2}(1+q).$$

From this we conclude that an invariant trace τ on $A=Z_p(G_p)$ must assume the same value on each q^ν, $0\leqq\nu\leqq p-1$, and as such must be a multiple of the augmentation $\varepsilon:Z_p(G_p)\to Z_p$. The kernel of ε being an ideal, ε is a degenerate trace and so is τ.

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