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ALTERNATIVE RINGS WITHOUT NILPOTENT ELEMENTS

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ABSTRACT. In this paper we show that any alternative ring without nonzero nilpotent elements is a subdirect sum of alternative rings without zero divisors.

Andrunakievic and Rjabuhin proved the corresponding result for associative rings by a complicated[1] process in 1968. Our result extends Andrunakievic and Rjabuhin's result to the alternative case, and our argument is nearly as simple as in the associative-commutative case. Since right alternative rings of characteristic not 2 without nilpotent elements are alternative, our results extend to such rings as well.

Introduction. A nonassociative ring is said to be without zero divisors if $ab=0$ implies $a=0$ or $b=0$. In a power associative ring, an element a is called nilpotent if $a^n=0$ for some positive integer n. The direct sum of rings without zero divisors will have zero divisors. The property which carries over into the direct sum is that there are no nonzero nilpotent elements. In the alternative case, either of these properties characterizes such rings.

THEOREM. *Let A be an alternative ring. The following are equivalent:*
(a) *A has no nonzero nilpotent elements.*
(b) *A is a subdirect sum of rings without zero divisors.*

This result is well-known for associative-commutative rings. It was extended to associative rings by Andrunakievic and Rjabuhin in 1968 [**1**]. Our result shows that the correspondence is even more fundamental, and our argument is nearly as simple as in the associative-commutative case. The characterization even extends to right alternative rings of characteristic $\neq 2$, since Kleinfeld [**4**] has shown that right alternative rings without nonzero nilpotent elements are alternative.

General remarks. Throughout the paper, A will be an alternative ring with no nonzero nilpotent elements. We base our proof on two well-known properties of alternative rings, both of which are proven in [**5**].

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[1] A simplified proof of Andrunakievic and Rjabuhin's result [**1**] has recently been obtained by A. Abian.

(a) (Artin) Any subring of A generated by two elements is associative.

(b) (Moufang) $a(xy)a=(ax)(ya)$ for all elements a, x, y in A.

By a product we mean a finite set of elements of the ring $a_1, a_2, a_3, \cdots, a_n$ multiplied together in some association. Each a_i is called a factor of the product. When we say a rearrangement, we mean any other possible product formed by multiplying the same elements together. The length of the product is the number of factors involved.

Main section. The difficulties are contained in the following lemma.

LEMMA 1. *If A is an alternative ring with no nonzero nilpotent elements, then*:

(a) $ab=0$ *implies* $ba=0$.

(b) $(ab)c=0 \Leftrightarrow a(bc)=0$.

(c) *If a product having x as a factor is zero, the product is still zero if x is replaced by any element in the ideal generated by x.*

(d) *If a product of elements is zero in some arrangement, then the product is zero for all other arrangements.*

PROOF OF (a). $ab=0$ implies $(ba)^2=b(ab)a=0$. By hypothesis, $ba=0$.

PROOF OF (b). If $(ab)c=0$, then, using the Moufang identity,

$$\begin{aligned}(a(bc))^3 &= a(bc\cdot(a\cdot bc\cdot a)\cdot bc) = a(bc\cdot(ab\cdot ca)\cdot bc)\\ &= a((bc\cdot ab)(ca\cdot bc)) = a((bc\cdot ab)(c\cdot ab\cdot c)) = 0.\end{aligned}$$

By hypothesis, $a(bc)=0$. The reverse implication is proved similarly.

PROOF OF (c). If $ax=0$ and r is any element of A, then $0=(ax)r=a(xr)$ and $0=ax=xa=r(xa)=(rx)a=a(rx)$. By repeated application of this argument, we get $a\langle x\rangle=0$ where $\langle x\rangle$ is the ideal generated by x. Similarly, if $xa=0$, then $\langle x\rangle a=0$. If a product w involves a factor x, let $w\#$ represent the same product, with the exception that x is replaced by some $i\in\langle x\rangle$. We have shown that $0=w=ax$ implies $w\#=0$. Similarly, $0=w=xa$ implies $w\#=0$. If $w\neq x$, then $w=ab$, and we use induction on the length of the product a or b involving x. If length is 1, then clearly $a=x$, or $b=x$. This case has been shown above. Assume the result for all shorter products. We shall now show the specific case that x is in a, and $a=cd$ with x in d. All other cases are similar. The procedure is as follows:

$$0 = w = ab = (cd)b = b(cd) = (bc)d = (bc)d\# = b(cd\#) = (cd\#)b = w\#.$$

PROOF OF (d). Suppose w is a product of $a_1, a_2, a_3, \cdots, a_n$ in some association. Let w' be any rearrangement. $w'\in\langle a_i\rangle$ for each i. Furthermore, replacing each a_i in w by w', using part (c), we have $(w')^n=0$. By hypothesis, $w'=0$.

The fundamental idea in this proof was to examine commutativity, not in general (i.e. between any elements of the ring) but rather in the case of rearrangements of factors in products which are zero. This idea was taken from Andrunakievic and Rjabuhin's paper [**1**].

We now return to the proof of the theorem. Let us call a subset M of A a multiplicative system if

(1) $0 \notin M$, and

(2) $m, m' \in M$ implies $mm' \in M$.

For each $a \in A$, $a \neq 0$, the set $M_a = \{a^i | i = 1, 2, \cdots\}$ is a multiplicative system. By a Zorn's Lemma argument, there exists a maximal multiplicative system containing a. In the following Lemma we show that the complement of a maximal multiplicative system M, $c(M)$, is an ideal. Then $A/c(M)$ is without zero divisors. The standard mapping of A onto the subdirect sum of the rings A/I, as I runs over all ideals of A such that A/I is without zero divisors, shows that A is a subdirect sum of rings without zero divisors. The map is evidently 1-1, since each nonzero element a is contained in some maximal multiplicative system.

LEMMA 2. *If M is a maximal multiplicative system, then* $c(M) = \{x \in A | x \notin M\}$ *is an ideal.*

PROOF. If $x \in c(M)$, then by the maximality of M, there exists a product w, consisting of x's and elements of M, which is zero. Since $w\# = 0$, we have $\langle x \rangle \subseteq c(M)$. This shows that the multiplicative property for an ideal holds. We now show $c(M)$ is closed under subtraction. If x and y are in $c(M)$, there exists a product, consisting of x's and elements of M, which is zero. By part (d) of Lemma 1, we have $x^i m = 0$ for some $i \geqq 1$ and $m \in M$. Since $(xm)^i$ is a rearrangement of $x^i m \cdot m^{i-1} = 0$, we get $(xm)^i = 0$; thus, by hypothesis, $xm = 0$. For y there also exists $m' \in M$ such that $ym' = 0$. Then $0 = xm = mx = mxm' = mym' = m(x-y)m'$. Thus $x - y \notin M$. This completes the proof.

We have completed the proof of the theorem. For those readers who are interested in the summands, the associative rings without zero divisors are covered in [**2**]. The alternative rings which are not associative are covered in [**3**].

Generalization. The only property used in the proof is property (d) of Lemma 1. Furthermore, (d) automatically holds in any ring which is a direct sum of rings without zero divisors. We state as a corollary to the proof of the theorem:

COROLLARY. *A nonassociative ring R is a subdirect sum of rings without zero divisors $\Leftrightarrow R$ has no nonzero nilpotent elements and R satisfies* (d).

Jordan rings provide an example of nonassociative rings where the theorem is false. Let Q be the quaternions with the standard basis $1, i, j, k$ over the reals. Q under the Jordan product $a^0b=\frac{1}{2}(ab+ba)$ is a simple Jordan ring and hence is subdirectly irreducible. $(Q, +, ^0)$ has no nilpotent elements, but does have zero divisors as, for example, $i^0j=\frac{1}{2}(ij+ji)=0$.

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