

ON THE CONTINUITY OF FUNCTIONS IN W_p^1 WHICH ARE MONOTONIC IN ONE DIRECTION

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ABSTRACT. It was previously shown that, for $n=2$, if f is such that its distribution derivatives are measures, and f is monotonically nondecreasing in one variable for almost all values of the other variable then f is equivalent to a continuous function. This is now shown to be false for $n>2$. It is true for $f \in W_p^1$, $p>n-1$ and may be false for $f \in W_{n-1}^1$.

In a previous note [1] we showed that there are functions f of n variables which are in W_1^1 and are such that, for each direction d , f does not have any decomposition $f=g-h$, with g and h monotonically nondecreasing in direction d and belonging to W_1^1 . For the proof, we used the fact, also proved in [1], that for $n=2$ a function of this sort is equivalent to a function which is continuous almost everywhere. At that time we did not know whether this latter fact holds for $n>2$.

The purpose of this note is to completely settle the matter. For $n>2$, there is an $f \in W_{n-1}^1$, with the above monotonicity property, which is discontinuous everywhere, but each $f \in W_p^1$, $p>n-1$, with this monotonicity property, is equivalent to a function continuous almost everywhere.

Let $n>2$. Let (x_1, x_2, \dots, x_n) be cartesian coordinates, and for each $i=1, \dots, n$, write (x_1, \dots, x_n) as (x_i, \bar{x}_i) . Then \bar{x}_i denotes the point $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ in $(n-1)$ -space. It is well known that for functions of m variables, $m \geq 2$, there are $f \in W_m^1$ such that each g equivalent to f is discontinuous everywhere. An easy construction is given in [3]. Let $g(\bar{x}_i)$ be such a function. Then $g \in W_{n-1}^1$ and the function $f(x)=f(x_i, \bar{x}_i)=g(\bar{x}_i)$ is monotonically nondecreasing in x_i for all values of \bar{x}_i and is such that every function equivalent to f is everywhere discontinuous.

However, as we now show, if $f \in W_p^1$, $p>n-1$, has the above monotonicity property, there is a g , equivalent to f , which is continuous almost everywhere. The proof follows from a fact, obtained in [2], that if $f \in W_p^1$, $p>n-1$, there is an equivalent g such that for each $i=1, \dots, n$, g is

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continuous in \bar{x}_i for almost all values of x_i . Moreover, g is continuous in x_i for almost all \bar{x}_i .

Let the domain of g be a cube, which we may assume to be closed. Suppose the set of discontinuities of g has positive measure. Let R be the $(n-1)$ -cube whose edges have the same lengths as those of the domain of g .

There is $k > 0$ such that the saltus of g exceeds k for each point in a measurable set H with the following property: There is a linear set E of 1-measure greater than k such that, for each $x_i \in E$, the set of $\bar{x}_i \in R$ for which $(x_i, \bar{x}_i) \in H$ has $(n-1)$ -measure greater than k . We may suppose g continuous on R for each $x_i \in E$ and further that there is a partition R_1, \dots, R_q of R into equal cubes such that the oscillation in \bar{x}_i of $g(x_i, \bar{x}_i)$, on each R_j , is less than $k/3$ for each $x_i \in E$. There is $\delta > 0$ and $F \subset R$ such that $|F| > |R| - |R_j|$, and for each $\bar{x}_i \in F$, $|u_i - v_i| < \delta$ implies $|g(u_i, \bar{x}_i) - g(v_i, \bar{x}_i)| < k/3$. There are $x_i \in E$, $y_i \in E$, $z_i \in E$ with $x_i < y_i < z_i$ and with $y_i - x_i < \delta$ and $z_i - y_i < \delta$. There is a $\bar{u}_i \in \text{int } R_j$, for some j , such that $(y_i, \bar{u}_i) \in H$. So, there is a (t_i, \bar{v}_i) with $\bar{v}_i \in R_j$ and either $x_i < t_i < y_i$ or $y_i < t_i < z_i$, for which $|g(t_i, \bar{v}_i) - g(y_i, \bar{u}_i)| > k$. We consider the case $y_i < t_i < z_i$ and $g(t_i, \bar{v}_i) - g(y_i, \bar{u}_i) > k$. Since $g(z_i, \bar{v}_i) \geq g(t_i, \bar{v}_i)$, we have $g(z_i, \bar{v}_i) - g(y_i, \bar{u}_i) > k$. But $F \cap R_j \neq \emptyset$, so there is $\bar{w}_i \in R_j$ such that $g(z_i, \bar{w}_i) - g(y_i, \bar{w}_i) < k/3$. We now have

$$\begin{aligned} g(z_i, \bar{v}_i) - g(y_i, \bar{u}_i) &= [g(z_i, \bar{v}_i) - g(z_i, \bar{w}_i)] + [g(z_i, \bar{w}_i) - g(y_i, \bar{w}_i)] \\ &\quad + [g(y_i, \bar{w}_i) - g(y_i, \bar{u}_i)] \\ &< k/3 + k/3 + k/3 = k. \end{aligned}$$

The case $x_i < t_i < y_i$ and $g(t_i, \bar{v}_i) - g(y_i, \bar{u}_i) < -k$ is similar, and the other two cases are easier. This contradicts the assumption that g is discontinuous on a set of positive measure, and we have the following result.

THEOREM. *If f is a function of $n \geq 3$ variables, $f \in W_p^1$, $p > n-1$, and f is monotonically nondecreasing in one variable for almost all values of the other variables, then f is equivalent to a function which is continuous almost everywhere. The statement is false for $p = n-1$.*

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