PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 2, February 1974

# ON THE CONTINUITY OF FUNCTIONS IN $W_p^1$ WHICH ARE MONOTONIC IN ONE DIRECTION

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ABSTRACT. It was previously shown that, for $n=2$, if $f$ is such that its distribution derivatives are measures, and $f$ is monotonically nondecreasing in one variable for almost all values of the other variable then $f$ is equivalent to a continuous function. This is now shown to be false for $n>2$. It is true for $f\in W_p^1$, $p>n-1$ and may be false for $f\in W_{n-1}^1$.

In a previous note [1] we showed that there are functions $f$ of $n$ variables which are in $W_1^1$ and are such that, for each direction $d$, $f$ does not have any decomposition $f=g-h$, with $g$ and $h$ monotonically nondecreasing in direction $d$ and belonging to $W_1^1$. For the proof, we used the fact, also proved in [1], that for $n=2$ a function of this sort is equivalent to a function which is continuous almost everywhere. At that time we did not know whether this latter fact holds for $n>2$.

The purpose of this note is to completely settle the matter. For $n>2$, there is an $f\in W_{n-1}^1$, with the above monotonicity property, which is discontinuous everywhere, but each $f\in W_p^1$, $p>n-1$, with this monotonicity property, is equivalent to a function continuous almost everywhere.

Let $n>2$. Let $(x_1, x_2, \cdots, x_n)$ be cartesian coordinates, and for each $i=1,\cdots,n$, write $(x_1,\cdots,x_n)$ as $(x_i, \bar{x}_i)$. Then $\bar{x}_i$ denotes the point $(x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n)$ in $(n-1)$-space. It is well known that for functions of $m$ variables, $m\geqq 2$, there are $f\in W_m^1$ such that each $g$ equivalent to $f$ is discontinuous everywhere. An easy construction is given in [3]. Let $g(\bar{x}_i)$ be such a function. Then $g\in W_{n-1}^1$ and the function $f(x)=f(x_i,\bar{x}_i)=g(\bar{x}_i)$ is monotonically nondecreasing in $x_i$ for all values of $\bar{x}_i$ and is such that every function equivalent to $f$ is everywhere discontinuous.

However, as we now show, if $f\in W_p^1$, $p>n-1$, has the above monotonicity property, there is a $g$, equivalent to $f$, which is continuous almost everywhere. The proof follows from a fact, obtained in [2], that if $f\in W_p^1$, $p>n-1$, there is an equivalent $g$ such that for each $i=1,\cdots,n$, $g$ is

Received by the editors March 6, 1973.
*AMS (MOS) subject classifications* (1970). Primary 26A15, 46E35.
[1] Supported by a grant from the National Science Foundation.

continuous in $\bar{x}_i$ for almost all values of $x_i$. Moreover, $g$ is continuous in $x_i$ for almost all $\bar{x}_i$.

Let the domain of $g$ be a cube, which we may assume to be closed. Suppose the set of discontinuities of $g$ has positive measure. Let $R$ be the $(n-1)$-cube whose edges have the same lengths as those of the domain of $g$.

There is $k>0$ such that the saltus of $g$ exceeds $k$ for each point in a measurable set $H$ with the following property: There is a linear set $E$ of 1-measure greater than $k$ such that, for each $x_i \in E$, the set of $\bar{x}_i \in R$ for which $(x_i, \bar{x}_i) \in H$ has $(n-1)$-measure greater than $k$. We may suppose $g$ continuous on $R$ for each $x_i \in E$ and further that there is a partition $R_1, \cdots, R_q$ of $R$ into equal cubes such that the oscillation in $\bar{x}_i$ of $g(x_i, \bar{x}_i)$, on each $R_j$, is less than $k/3$ for each $x_i \in E$. There is $\delta>0$ and $F \subset R$ such that $|F|>|R|-|R_j|$, and for each $\bar{x}_i \in F$, $|u_i - v_i|<\delta$ implies $|g(u_i, \bar{x}_i)-g(v_i, \bar{x}_i)|<k/3$. There are $x_i \in E$, $y_i \in E$, $z_i \in E$ with $x_i<y_i<z_i$ and with $y_i - x_i<\delta$ and $z_i - y_i<\delta$. There is a $\bar{u}_i \in \operatorname{int} R_j$, for some $j$, such that $(y_i, \bar{u}_i) \in H$. So, there is a $(t_i, \bar{v}_i)$ with $\bar{v}_i \in R_j$ and either $x_i<t_i<y_i$ or $y_i<t_i<z_i$, for which $|g(t_i, \bar{v}_i)-g(y_i, \bar{u}_i)|>k$. We consider the case $y_i<t_i<z_i$ and $g(t_i, \bar{v}_i)-g(y_i, \bar{u}_i)>k$. Since $g(z_i, \bar{v}_i) \geqq g(t_i, \bar{v}_i)$, we have $g(z_i, \bar{v}_i)-g(y_i, \bar{u}_i)>k$. But $F \cap R_j \neq \varnothing$, so there is $\bar{w}_i \in R_j$ such that $g(z_i, \bar{w}_i)-g(y_i, \bar{w}_i)<k/3$. We now have

$$\begin{aligned} g(z_i, \bar{v}_i) - g(y_i, \bar{u}_i) = {} & [g(z_i, \bar{v}_i) - g(z_i, \bar{w}_i)] + [g(z_i, \bar{w}_i) - g(y_i, w_i)] \\ & + [g(y_i, \bar{w}_i) - g(y_i, \bar{u}_i)] \\ & < k/3 + k/3 + k/3 = k. \end{aligned}$$

The case $x_i<t_i<y_i$ and $g(t_i, \bar{v}_i)-g(y_i, \bar{u}_i)<-k$ is similar, and the other two cases are easier. This contradicts the assumption that $g$ is discontinuous on a set of positive measure, and we have the following result.

THEOREM. *If $f$ is a function of $n \geqq 3$ variables, $f \in W_p^1$, $p>n-1$, and $f$ is monotonically nondecreasing in one variable for almost all values of the other variables, then $f$ is equivalent to a function which is continuous almost everywhere. The statement is false for $p=n-1$.*

## REFERENCES


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