PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 2, February 1974

RATIONAL APPROXIMATION OF EXTREMAL LENGTH FOR DOUBLY CONNECTED DOMAINS

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ABSTRACT. Results on the approximation of analytic functions are used to approximate the extremal length of the family of curves separating the boundary components of a doubly connected domain. Bounds for the approximations are established.

1. **Introduction.** In this report we will consider the problem of approximating the extremal length of the family of curves separating the boundary components of a doubly connected domain. A method of successive approximations will be developed using a basis for the Hilbert space of square-integrable analytic functions on the domain. It will be shown that the extremal length can be obtained from the solution of a minimization problem in this infinite-dimensional space and, furthermore, that this solution can be approximated by solving a minimization problem in a finite-dimensional subspace. In the problem for the whole space there are an infinite number of constraints, but in the problem for the subspace there are only a finite number of constraints.

The method of approximation which will be developed here is somewhat similar to that of the approximation of the solution of the minimum problem associated with the Bergman kernel function. That method may be found in the book by Kantorovich and Krylov [2].

2. **The extremal length and module of a doubly connected domain.** Let D be a bounded doubly connected domain bounded by analytic curves and let Γ be a family of simple closed contours in D which separate the boundary components of D. The module of Γ is defined to be

$$m(\Gamma) = \inf\left\{ \iint_D \rho^2 \mid \rho \in U \right\}$$

where U is the set of all nonnegative measurable functions on D such that, for each γ in Γ, ρ is a measurable function of arc length on γ and the integral with respect to arc length $\int_\gamma \rho \geqq 1$. The extremal length of Γ is $\lambda(\Gamma) = [m(\Gamma)]^{-1}$.

Received by the editors September 29, 1972 and, in revised form, February 15, 1973 and May 14, 1973.

AMS (MOS) subject classifications (1970). Primary 30A82, 31A15.

Let C_0 be the family of all simple closed contours separating the boundary components of D. We will call $m(C_0)$ the module of D. If F is a one-to-one conformal mapping of D onto the circular ring $1<|z|<r$, then

$$m(C_0) = \iint_D \rho^2 \quad \text{and} \quad \int_c \rho \geqq 1$$

for every c in C_0 if and only if $\rho=|F'|/2\pi|F|$, except possibly on a set of measure zero. We note that $m(C_0)=(1/2\pi)\log r$. The proofs of the above-mentioned results and other properties of module and extremal length may be found in the book by Jenkins [**1**].

Let $L^2(D)$ be the Hilbert space of square-integrable analytic functions on D with inner product

$$(f, g) = \iint_D f\bar{g}.$$

The module of D can now be characterized as

$$m(C_0) = \inf\left\{(f,f) \,\middle|\, f \in L^2(D) \text{ and } \int_c |f| \geqq 1 \text{ for every } c \text{ in } C_0\right\}.$$

As has been seen, the function $F'/2\pi F$ gives the desired minimum value. If f and g are two minimizing functions, then $|f|=|g|$ and hence $f=e^{i\theta}g$ for some real number θ.

3. **Approximating the module by families of curves with bounded length.** Let K be a positive constant greater than the length of either boundary component of D. For each such K we define the family $C(K)$ consisting of the curves in C_0 with length less than or equal to K.

THEOREM 3.1. *There is a constant $K_0>0$ such that $m(C(K))=m(C_0)$ for all $K \geqq K_0$.*

PROOF. Let R be the circular ring $1<|z|<r$ which is conformally equivalent to the domain D. Let Γ be the collection of circles given by $z=\rho e^{i\theta}$ where $1<\rho<r$. Since the module of R is $(1/2\pi)\log r$, $m(\Gamma)\leqq (1/2\pi)\log r$. It can be readily shown, as in Jenkins [**1**, p. 18], that $f(z)=2\pi/z$ possesses the properties

$$\iint_R |f|^2 = \frac{1}{2\pi}\log r \quad \text{and} \quad \int_\gamma |f| \geqq 1 \quad \text{for } \gamma \in \Gamma.$$

Now let f be any function in $L^2(R)$ such that $\int_\gamma |f| \geqq 1$ for $\gamma \in \Gamma$. Then since $\int_\gamma |f| = \rho \int_0^{2\pi} |f(\rho e^{i\theta})| \, d\theta$, as in Jenkins [1, p. 18],

$$\frac{1}{2\pi}\int_0^{2\pi}\int_1^r |f(\rho e^{i\theta})| \, d\rho \, d\theta \geqq \frac{1}{2\pi}\log r = \frac{1}{4\pi^2}\int_0^{2\pi}\int_1^r \frac{1}{\rho}\, d\rho\, d\theta,$$

which implies

$$\begin{aligned} 0 &\leqq \int_0^{2\pi}\int_1^r \left(|f(\rho e^{i\theta})| - \frac{1}{2\pi\rho}\right)^2 \rho\, d\rho\, d\theta \\ &\leqq \int_0^{2\pi}\int_1^r |f(\rho e^{i\theta})|^2 \rho\, d\rho\, d\theta - \frac{1}{\pi}\int_0^{2\pi}\int_1^r |f(\rho e^{i\theta})|\, d\rho\, d\theta + \frac{1}{4\pi^2}\int_0^{2\pi}\int_1^r \frac{1}{\rho}\, d\rho\, d\theta \\ &\leqq \int_0^{2\pi}\int_1^r |f(\rho e^{i\theta})|^2 \rho\, d\rho\, d\theta - \frac{1}{2\pi}\log r. \end{aligned}$$

Thus

$$\iint_R |f|^2 \geqq \frac{1}{2\pi}\log r \quad \text{and} \quad m(\Gamma) = \frac{1}{2\pi}\log r.$$

Let F be a one-to-one conformal mapping of R onto D. Let Λ be the image of Γ under F. Then $m(\Lambda) = m(\Gamma)$ which implies $m(C_0)$, the module of D, is equal to $m(\Lambda)$ since $m(\Gamma)$ is equal to the module of R. For any λ in Λ there is a γ in Γ such that $f(\gamma) = \lambda$. Hence the length of λ, which is $\int_\gamma |F'|$, satisfies

$$\int_\gamma |F'| \leqq 2\pi r M$$

where $M = \max\{|F'(z)| \mid z \in R\}$. Let $K_0 = 2\pi r M$. Then $m(\Lambda) \leqq m(C(K)) \leqq m(C_0)$ for any $K \geqq K_0$ and since $m(\Lambda) = m(C_0)$, $m(C(K)) = m(C_0)$ for $K \geqq K_0$.

In the proof of the above theorem, a value for K_0 is found which depends on the conformal mapping function. Assuming this function is not known, we do not know exactly how large K_0 must be. It is sufficient that K_0 be larger than the maximum length of the images of the circles in Γ. If the original domain D were the circular ring $1 < |z| < r$, then $m(C(K)) = m(C_0)$ for all K under consideration; that is, $K > 2\pi r$.

4. **Approximating the module for curves with bounded length by rational functions.** In the remaining part of this paper, we assume that K is large enough so that $m(C(K)) = m(C_0)$. The family $C(K)$ will be denoted simply by C. This section deals with approximating $m(C)$ which is equal to the module of D.

Let $\{C_n\}$ be a sequence of finite families of curves such that

$$C_1 \subset C_2 \subset \cdots \subset C$$

and, for any given c in C, $c:z=\varphi(t)$, $0\leqq t\leqq 1$, and $\varepsilon>0$, there exists an $N>0$ so that whenever $n>N$ we can find a c_n in C_n, $c_n:z=\varphi_n(t)$, $0\leqq t\leqq 1$, with $|\varphi_n(t)-\varphi(t)|<\varepsilon$ for $0\leqq t\leqq 1$ and $|\varphi_n'(t)-\varphi'(t)|<\varepsilon$ if $\varphi_n'(t)$ and $\varphi'(t)$ are both defined. The following is an example of such a sequence. For each positive integer n, let P_n be a finite set of points in D with $P_1\subset P_2\subset\cdots$ and $\bigcup_{n=1}^{\infty} P_n$ dense in D. If C is the family of simple closed polygonal paths contained in C with vertices in P_n, then $\{C_n\}$ satisfies the above conditions.

Let $\{w_k\}$ be a basis for $L^2(D)$ consisting of rational functions with all poles at a single point $z=\alpha$ in the bounded component of the complement of the closure of D (see Meschkowski [3, p. 83]). We will be concerned with the relation between the following quantities and the module of D:

$$m(C_n) = \inf\left\{(f,f)\,\middle|\, f\in L^2(D) \text{ and } \int_c |f| \geqq 1 \text{ for } c \text{ in } C_n\right\},$$

$$m_k(C) = \inf\left\{(f,f)\,\middle|\, f=\sum_{i=1}^{k} a_i w_i \text{ and } \int_c |f| \geqq 1 \text{ for } c \text{ in } C\right\},$$

$$m_k(C_n) = \inf\left\{(f,f)\,\middle|\, f=\sum_{i=1}^{k} a_i w_i \text{ and } \int_c |f| \geqq 1 \text{ for } c \text{ in } C_n\right\}.$$

Each of the above is a positive finite quantity and in each case the minimum value is assumed. From the definitions we have $m(C_n)\leqq m(C)\leqq m_k(C)$, $m(C_n)\leqq m_k(C_n)\leqq m_k(C)$.

THEOREM 4.1. $\lim_{k\to\infty} m_k(C)=m(C)$.

PROOF. We have already noted that $m(C)\leqq m_k(C)$ for any k. Suppose $m_k(C)$ does not converge to $m(C)$. Since $\{m_k(C)\}$ is a monotone sequence, there is an $\varepsilon>0$ such that $m_k(C)>m(C)+\varepsilon$ for every k.

Let f be the minimizing function $F'/2\pi F$ as given in §2. Then since $C\subset C_0$ and $m(C)=m(C_0)$, we have a function f in $L^2(D)$ with $\int_c |f|\geqq 1$ for every c in C and $(f,f)=m(C)$. From the relation between f and the conformal mapping function given in §2 and the fact that D is bounded by analytic curves, we observe that f is analytic on the closure of D. Thus f can be expanded in a series $\sum_{i=1}^{\infty} a_i w_i$ which converges uniformly to f on D. Let $f_n=\sum_{i=1}^{n} a_i w_i$. Let δ be a positive number less than $1/K$ and N_δ be a positive integer such that $|f_n-f|<\delta$ whenever $n>N_\delta$. Let A denote the area of D and let M be a bound for $\{f_n\}$ on D. Then

$$|(f_n,f_n)-(f,f)| \leqq 2M\iint_D |f_n-f| < 2MA\delta$$

for $n > N_\delta$. Also

$$1 - \int_c |f_n| \leqq \int_c |f| - \int_c |f_n| \leqq \int_c |f - f_n| \leqq K\delta$$

for c in C and $n > N_\delta$. Thus $\int_c |f_n| \geqq 1 - K\delta > 0$. Let $g_n = (1-K\delta)^{-1} f_n$. Now

$$\int_c |g_n| \geqq 1$$

and

$$\begin{aligned}(g_n, g_n) - (f_n, f_n) &= [(1 - K\delta)^{-2} - 1](f_n, f_n) \\ &< [(1 - K\delta)^{-2} - 1][(f, f) + 2AM\delta].\end{aligned}$$

Hence, for $n > N_\delta$,

$$|(g_n, g_n) - (f, f)| \leqq |(g_n, g_n) - (f_n, f_n)| + |(f_n, f_n) - (f, f)| < \eta$$

where $\eta = 2AM\delta + [(1-K\delta)^{-2} - 1][(f,f) + 2AM\delta] > 0$. Choose δ small enough so that $\eta < \varepsilon/2$. Then for $n > N_\delta$,

$$m_n(C) > m(C) + \varepsilon = (f, f) + \varepsilon > (g_n, g_n) + \varepsilon/2.$$

This contradicts the definition of $m_n(C)$.

THEOREM 4.2. $\lim_{n\to\infty} m(C_n) = m(C)$.

PROOF. For each positive integer n there is a function f_n in $L^2(D)$ such that $(f_n, f_n) = m(C_n)$ and $\int_c |f_n| \geqq 1$ for every c in C_n. Since $m(C_n) \leqq m(C)$ for each n, the sequence $\{(f_n, f_n)\}$ is bounded. This implies that the sequence of functions $\{f_n\}$ is uniformly bounded on compact subsets of D and hence is a normal family. Now there is a subsequence $\{f_{n_p}\}$ of $\{f_n\}$ which converges to a function f in $L^2(D)$ uniformly on compact subsets of D. By considering an exhaustion of D by compact subsets, it can be shown that

$$\liminf_{p\to\infty} (f_{n_p}, f_{n_p}) \geqq (f, f).$$

Also, from the definition of C_n, it is evident that $\int_c |f| \geqq 1$ for every c in C. Thus $(f, f) \geqq m(C)$ which implies

$$\liminf_{p\to\infty} m(C_{n_p}) \geqq m(C).$$

Since $m(C_n) \leqq m(C)$ and $\{m(C_n)\}$ is a monotone sequence, we have

$$\lim_{n\to\infty} m(C_n) = m(C).$$

THEOREM 4.3. $\lim_{n,k\to\infty} m_k(C_n) = m(C)$.

PROOF. By Theorems 4.1 and 4.2,

$$m(C) = \lim_{k\to\infty} m_k(C) = \lim_{n\to\infty} m(C_n).$$

The result now follows since $m(C_n) \leqq m_k(C_n) \leqq m_k(C)$.

5. **Bounds for** $m(C)$ **and** $m_k(C_n)$. An upper bound for $M(C)$ is first obtained involving the Bergman kernel function for the domain D. Let ζ be an arbitrary point in D and let f_ζ be the unique function with minimum norm from among all functions in $L^2(D)$ assuming the value 1 at $z=\zeta$. This function is related to the Bergman kernel function $K(z, \zeta)$ by the equation

$$f_\zeta(z) = K(z, \zeta)/K(\zeta, \zeta).$$

Since D is doubly connected, $K(z, \zeta)$ does not have a single-valued integral and hence $\int_c K(z, \zeta)\, dz \neq 0$ for $c \in C$. We also note that $K(\zeta, \zeta) > 0$. The verification of the above results are found in the book by Meschkowski [3, Chapter 4].

Since, for all c in C,

$$\int_c |f_\zeta| \geqq \left| \int_c f_\zeta(z)\, dz \right| = \frac{1}{K(\zeta, \zeta)} \left| \int_c K(z, \zeta)\, dz \right| > 0$$

and the last quantity is constant for c in C, we have

$$I = \inf\left\{ \int_c |f_\zeta| \;\middle|\; c \in C \right\} > 0.$$

Now $\int_c I^{-1}|f_\zeta| \geqq 1$, which implies $I^{-2} \iint_D |f_\zeta|^2 \geqq m(C)$. If we let

$$J = \left| \int_c f_\zeta(z)\, dz \right|,$$

then since $J \leqq I$ and

$$\iint_D |f_\zeta|^2 = 1/K(\zeta, \zeta),$$

it follows that

$$1/J^2 K(\zeta, \zeta) \geqq 1/I^2 K(\zeta, \zeta) \geqq m(C).$$

Thus the module $m(C)$ of D satisfies the inequality

$$\frac{1}{m(C)} \geqq J^2 K(\zeta, \zeta) = \frac{1}{K(\zeta, \zeta)} \left| \int_c K(z, \zeta)\, dz \right|^2$$

where c is an arbitrary curve in C.

A similar result for $m_k(C_n)$ can be obtained by using the reproducing kernel $K_k(z, \zeta)$ for the subspace of $L^2(D)$ generated by the basis elements

$w_1, \cdots, w_k$. In this case we have

$$\frac{1}{m_k(C_n)} \geqq \frac{1}{K_k(\zeta, \zeta)} \left| \int_c K_k(z, \zeta)\, dz \right|^2$$

where c is an arbitrary curve in C.

Since it is known that $K(z, \zeta)$ and $K_k(z, \zeta)$ are analytic functions of z in the closure of D whenever ζ is fixed in D, in each of the two inequalities we can also let c be either of the boundary components of D.

In order to obtain a lower bound for $m_k(C_n)$, it will be assumed that the basis $\{w_k\}$ for $L^2(D)$ has been orthonormalized. Then

$$f = \sum_{i=1}^{k} a_i w_i$$

implies

$$(f, f) = \sum_{i=1}^{k} |a_i|^2.$$

Also

$$\int_c |f| \leqq \sum_{i=1}^{k} |a_i| \int_c |w_i| \quad \text{for } c \in C_n.$$

Let c_j, $1 \leqq j \leqq p_n$, denote the curves in C_n. If we put $A_i = |a_i|$ and $W_{ij} = \int_{c_j} |w_i|$, then there is a unique k-tuple $(A_1, \cdots, A_k)$ which minimizes $\sum_{i=1}^k A_i^2$ from among all k-tuples with $\sum_{i=1}^k A_i W_{ij} \geqq 1$, $1 \leqq j \leqq p_n$. Since $\int_{c_j} |f| \geqq 1$ implies $\sum_{i=1}^k A_i W_{ij} \geqq 1$, it follows that if $(A_1, \cdots, A_k)$ is a solution to this minimum problem, then

$$m_k(C_n) \geqq \sum_{i=1}^{k} A_i^2.$$

We have thus obtained upper and lower bounds for $m_k(C_n)$. In the first case the bound involves a reproducing kernel which can be obtained by solving a minimum problem with a unique solution. In the second case we again solve a different minimum problem with a unique solution.

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