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A NOTE ON MONOMIALS IN SEVERAL COMPLEX VARIABLES

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ABSTRACT. Monomials in C^n are characterized in the polydisk algebra $A(U^n)$ as functions whose modulus is constant on the distinguished boundary of U^n and whose zero set has an intersection with the diagonal of U^n consisting (at most) of the origin.

The note answers a question raised by F. Norguet in his course notes *Fonctions de plusieurs variables complexes*, Paris, 1971. The following characterization of monomials in the polydisk algebra is a generalization of a result of R. Bojanic and W. Stoll about a characterization of monomials among entire functions [**1**].

We use the following notations:

U^n is the open unit polydisk in C^n.

T^n is the distinguished boundary of U^n.

$\Delta^n=\{(\lambda, \cdots, \lambda) \in \bar{U}^n, \lambda \in \bar{U}^1\}$ is the diagonal of $\bar{U}^n$.

$A(U^n)$ is the algebra of functions analytic in U^n and continuous in $\bar{U}^n$. For $f \in A(U^n)$ we call Z_f the zero set of f in $\bar{U}^n$. We recall that if $f, g \in A(U^n)$ and $f|T^n=g|T^n$ then $f=g$.

THEOREM. *Assume that $f \in A(U^n)$ satisfies*

(i) $|f|T^n|=1$,

(ii) $Z_f \cap \Delta^n \subset \{(0, \cdots, 0)\}$.

Then $f=cz_1^{k_1} \cdots z_n^{k_n}$ where $c \in T^1$.

PROOF. Let $z \in T^n$. Then $f_z(\lambda)=f(\lambda z)$ is a finite Blaschke product in $A(U^1)$ since (i)$\Rightarrow f$ is an "inner function" in $A(U^n)$, hence a rational function [**2**, p. 112]. Hence $f_z(\lambda)=c(z)\lambda^{p(z)}$ by (ii), where $p(z)$ is nonnegative integer valued. For $\lambda=1$, $f(z)=c(z)$ $\forall z \in T^n$, hence $f_z(\lambda)=f(z)\lambda^{p(z)}$. This shows that $p(z)$ is continuous integer valued on T^n for fixed $\lambda \in \bar{U}^1$. Hence $p(z)=k$, a fixed nonnegative integer since T^n is connected.

It follows that $f(\lambda z)=\lambda^k f(z)$ $\forall z \in T^n$ hence also for $\forall z \in U^n$. As seen from the Taylor expansion of f, f must be a homogeneous polynomial

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of degree k. (This part of the proof parallels an argument of S. Bochner [3].) Denote by k_i the degree of f in z_i $(i=1, \cdots, n)$. Write $f(z)=Q_i(z)z_i^{k_i}$ plus terms of lower degree in z_i where $Q_i \neq 0$ is a polynomial that does not involve z_i. There is at least one point of T^n where all Q_i are $\neq 0$. (If some Q_i were zero at each point of T^n, then $\prod_i Q_i$ would be zero on T^n and hence identically zero on U^n. Hence one of the Q_i's would be identically zero—a contradiction.)

Assume such a point is $(1, 1, \cdots, 1)$; now $f(\lambda, 1, 1, \cdots, 1)$ is a Blaschke product hence it has all its k_1 zeros in U^1. An easy index computation (see [2, p. 89]) shows that $f(\lambda, \lambda, \cdots, \lambda)$ has degree $k_1 + \cdots + k_n$. It follows that f is a monomial, $f = cz_1^{k_1} \cdots z_n^{k_n}$ where $c \in T^1$.

Bibliography

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3. S. Bochner, *Entire functions in several variables with constant absolute values on a circular uniqueness set*, Proc. Amer. Math. Soc. **13** (1962), 117–120. MR **25** #4130.

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