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A NOTE ON DIVERGENCE-LIKE 2-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. The method given by Ford [1] for the existence and uniqueness of a solution in $H_0^1(I)$ for the boundary value problem $[h(x, x', t)]' = f(x, x', t)$, $x(0)=x(1)=0$ is shown to be a special case of Browder's method [3] for partial differential equations of generalized divergence form. Also it is shown that the solution of the above boundary value problem in $H_0^{1,p}(I)$ can be obtained under weaker hypotheses than those assumed by Ford.

In [1] Wayne T. Ford discussed the boundary value problem

$$(1) \qquad [h(x, x', t)]' = f(x, x', t), \qquad t \in I, x(0) = x(1) = 0,$$

arising in 1-dimensional variational problems. Existence and uniqueness of solutions in $H_0^1(I)$ were proved under the assumption

(a) $x(t) \to h(x(t), x'(t), t)$, $x(t) \to f(x(t), x'(t), t)$ are hemicontinuous maps from $H_0^1(I)$ to $L^2(I)$.

(b) There exist constants λ, η such that

$$\begin{aligned}(U - u)[f(U, V, t) - f(u, v, t)] + (V - v)[h(U, V, t) - h(u, v, t)] \\ \geqq \lambda(V - v)^2 - \eta(V - u)^2; \qquad \lambda > 0, \quad \lambda\pi^2 - \eta > 0,\end{aligned}$$

for all real u, v, U, V.

He showed that system (1) is equivalent to

$$(2) \qquad Zu = 0, \qquad u \in H_0^1(I)$$

where Z is the map from $H_0^1(I)$ to $H_0^1(I)$ uniquely defined by $Zu=z$, z being the solution of the boundary value problem

$$z'' - z = [h(u, u', t)]' - f(u, u', t), \qquad z(0) = z(1) = 0$$

(Z is explicitly constructed in [1, see Lemma 2.1 and 2.2]).

Under assumptions (a) and (b), Z is seen to be a strongly monotone hemicontinuous map. Strong monotonicity of Z implies at once that (2) can have no more than one solution [1, Theorem 3.1]. Ford then appeals to a theorem of Browder [2, pp. 18–24] on monotone operators, which proves the existence of a solution.

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We will show that the method described above is completely contained as a very special case in the method given by Browder [3] for solving partial differential equations of generalised divergence form:

$$A(u) = \sum_{|\alpha|\leqq m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(\mathscr{E}_m(u)(t), t)$$

where

$$D^{\alpha} = \prod_{j=1}^{\alpha} \left(\frac{\partial}{\partial t_j}\right)^{\alpha_j}, \qquad |\alpha| = \sum_{j=1}^{n} \alpha_j$$

and

$$\mathscr{E}_m u(t) = \{D^{\alpha}u(t) \mid |\alpha| \leqq m\}.$$

In fact, let $A(u)=f(u(t), u'(t), t)-dh(u(t), u'(t), t)/dt$. Following Browder we define "Dirichlet form" $\alpha(u, v)$:

$$\alpha(u, v) = (f(u, u', \cdot), v) + (h(u, u', \cdot), v')$$

where (,) denotes the inner product in $L^2(I)$. Then (1) is equivalent to the variational boundary value problem

$$(2') \qquad \alpha(u, v) = 0, \qquad u, v \in H_0^1(I).$$

By assumption (a), $\alpha(u, v)$ is defined for all $u \in H_0^1(I)$ and it is clearly a continuous functional of v in $H_0^1(I)$. It follows that there exists an element Zu in $H_0^1(I)$ such that

$$\alpha(u, v) = (Zu, v) + ((Zu)', v').$$

Thus (2′) is equivalent to $Zu=0$, $u \in H_0^1(I)$, which is (2). That Z is a hemicontinuous and strictly monotone map of $H_0^1(I)$ to $H_0^1(I)$ can be seen without having an explicit representation of Z. The hemicontinuity follows at once from (a). To check the strong monotonicity we use (b) to obtain

$$\begin{aligned}
&(Zu - Zv, u - v) + ((Zu)' - (Zu)', u' - v') \\
&\quad = \alpha(u, u - v) - \alpha(v, u - v) \\
&\quad = \int_I [f(u, u', \cdot) - f(v, v', \cdot)](u - v) + \int_I [h(u, u', \cdot) - h(u, u', \cdot)](u' - v') \\
&\quad \geqq \lambda \int_I (u' - v')^2 - \eta \int_I (u - v)^2 \geqq \min(\lambda, \lambda - \eta\pi^{-2}) \int_I (u' - v')^2 \\
&\quad \geqq K\left[\int_I (u - v)^2 + \int_I (u' - v')^2\right], \qquad K = (1 + \pi^{-2})^{-1/2} \min(\lambda, \lambda - \eta\pi^{-2}).
\end{aligned}$$

We add the following remarks.

1. Instead of (a) one can assume more explicit "Nemytskiĭ type" growth conditions on f and h as follows:

$$|f(u, v, t)| \leqq b(|u| + |v|) + a(t),$$
$$|h(u, v, t)| \leqq b(|u| + |v|) + a(t),$$

$b>0$, $a \in L^2(I)$, for all real u, v.

2. Let $H_0^{1,p}(I)$ $(1<p<\infty)$ be the space obtained by completing $C_0^\infty(I)$ in the norm $[\int_I (|u|^p+|u'|^p)]^{1/p}$. Following Browder [3], one concludes that (1) has a solution in $H_0^{1,p}(I)$ if the following conditions are satisfied:

$$\text{(a}'\text{)}\qquad \begin{aligned} |f(t, u, v)| &\leqq b(|u|^{p-1} + |v|^{p-1}) + a(t), \\ |h(u, v, t)| &\leqq b(|u|^{p-1} + |v|^{p-1}) + a(t), \end{aligned}$$

$b>0$, $a \in L^{p'}(I)$, $p'=p/(p-1)$, for all real u, v.

$$\text{(b}'\text{)(i)}\qquad \begin{aligned} &(U - u)[f(U, V, t) - f(u, v, t)] \\ &\qquad + (V - v)[h(U, V, t) - h(u, v, t)] \geqq 0 \end{aligned}$$

for all real u, v, U, V.

$$\text{(ii)}\qquad uf(u, v, t) + uh(u, v, t) \geqq \lambda |u|^p - \mu |v|^p + a_1(t),$$

$a_1 \in L^{p'}(I)$, $\lambda>0$, $\lambda-\mu>0$, for all sufficiently large $|u|+|v|$. If (b′) is replaced by

$$\text{(b}''\text{)}\qquad \begin{aligned} (U - u)[f(U, V, t) - f(u, v, t)] + (V - v)[h(U, V, t) - h(u, v, t)] \\ \geqq \lambda |U - u|^p - \mu |V - v|^p, \end{aligned}$$

$\lambda>0$, $\lambda-\mu>0$, for all real u, v, U, V, then (1) has a unique solution in $H_0^{1,p}(I)$.

3. If we appeal to Theorem 4, Browder [3], conditions (a′), (b′) may be replaced by the following ones:

$$\text{(a}'''\text{)}\qquad \begin{aligned} &|f(u, v, t)| \leqq a_1(t) + b |v|^{p-1}, \\ &|h(u, v, t)| \leqq a_2(t) + b |v|^{p-1}, \qquad a_1, a_2 \in L^{p'}, \; b > 0, \end{aligned}$$

$$\text{(b}'''\text{)(i)}\qquad (V - v)[h(u, V, t) - h(u, v, t)] \geqq 0 \quad \text{for all } u, v, V.$$

(ii) There exist constants $c_0>0$ and c_2 such that

$$uf(u, v, t) + vh(u, v, t) \geqq c_0[|u|^p + |v|^p] - c_2.$$

4. The above method applies with little change to the boundary value problem

$$\sum_{k=0}^{m} (-1)^k \frac{d^k}{dt^k} f_k(u(t), u'(t), \cdots, u^m(t), t) = 0, \qquad t \in I,$$
$$u^k(0) = u^k(1), \qquad k = 0, 1, \cdots, m - 1.$$

The condition that corresponds to (b) of the original assumption is in this case:

$$\sum_{k=0}^{m} (u_k - v_k)[f_k(u_0, u_1, \cdots, u_m, t) - f_k(v_0, v_1, \cdots, v_m, t)]$$
$$\geqq \sum_{k=0}^{m} \lambda_k(u_k - v_k), \qquad \lambda_m > 0, \ \lambda_m \pi^{2m} + \cdots + \lambda_0 > 0$$

for all real $u_0, \cdots, u_m, v_0, \cdots, v_m$.

The explicit construction of the operator corresponding to Z would be quite tedious in this case.

References

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