

THE NORMALITY OF CERTAIN SUBGROUPS OF ELEMENTARY SUBGROUPS OF STEINBERG GROUPS OVER RINGS

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ABSTRACT. This paper amends the approach used in an earlier paper to construct, from ideals in the Chevalley algebra L_R over a commutative ring R with identity, normal subgroups of the elementary subgroup G_R^1 of Steinberg's twisted group corresponding to L , a finite dimensional simple Lie algebra over the complex field. The set of normal subgroups so constructed turns out to be in one-to-one correspondence with the set of equivalence classes of ideals of R under an equivalence relation defined in terms of the underlying automorphism of R of order 2.

1. Introduction. In [3], to which the reader is referred for all notation not specified here, the same procedure developed in [2] was used with the aim of creating normal subgroups of G_R^1 corresponding to ideals I in the Chevalley algebra L_R of the Lie algebra L over R . R. W. Carter has pointed out to the author that the procedure of [2] fails in general to define a normal subgroup in the context of [3], since the elements $x_r(t)$ which lie in U_R^1 or V_R^1 do not generate the group G_R^1 . In the present paper, a modification of the approach found in [3, §4] is used to construct a normal subgroup G_I^1 of G_R^1 corresponding to an ideal I of L_R . The author would like to acknowledge and thank Professor Carter for the suggestion that a condition of the form $ue_s + \bar{u}e_{\bar{s}} \in I$ would be needed in the definition of G_I^1 (see (5) and (6) below).

While such a normal subgroup could be described as the subgroup of G_R^1 generated by a certain set of elements in the manner of [2], these elements are much more complicated than the corresponding generators of G_I in [2] so that such a description becomes quite unwieldy. The normal subgroup G_I^1 in [2] however is the normal closure of the subgroup generated by all elements $x_r(t)$ where $te_r \in I$. It is this fact which serves as the point of departure for the approach taken in the present paper. We define G_I to be the normal closure of the elements of type (4), (5), and

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(6) below. The problem, then, is to determine to what extent distinct ideals of L_R gives rise to distinct normal subgroups of G_R^1 . If $I \cap E_R = JE_R$ and $I' \cap E_R = J'E_R$, then we show that $G_I^1 = G_{I'}^1$ if and only if $J \cap \bar{J} = J' \cap \bar{J}'$. Here $E_R = R \otimes_Z E_Z$ where E_Z is the free abelian group on the Chevalley basis elements not in H . We thus obtain a bijective correspondence between the set of normal subgroups G_I^1 and the set of equivalence classes of ideals of R modulo the equivalence relation on the set of ideals of R defined by the condition $J' \cap \bar{J}' = J \cap \bar{J}$.

2. **Generators for G_R^1 .** We assume that the Lie algebra L has a symmetry $r \rightarrow \bar{r}$ of order 2 of its Coxeter–Dynkin diagram, so that L_R has a semiautomorphism of order 2 [6, Lemma 3.2]. We also assume that R has an automorphism $t \rightarrow \bar{t}$ of order 2. Then $\sigma: G_R \rightarrow G_R$ given by $\sigma(x_r(t)) = x_{\bar{r}}(\bar{t})$ is an automorphism. G_R^1 is the subgroup of G_R generated by U_R^1 and V_R^1 , which are the respective intersections of the fixed point group of σ with the maximal unipotent subgroups U_R and V_R generated by the elements $x_r(t)$ where r runs respectively over the positive and negative roots. Then G_R^1 is generated [6, Lemma 4.6] by elements of the forms:

- (1) $x_r(t)$, where $r = \bar{r}$, $t = \bar{t}$, $r \neq s + \bar{s}$ for any root s ;
- (2) $x_r(t)x_{\bar{r}}(\bar{t})$, where $r < \bar{r}$, and $r + \bar{r}$ is not a root;
- (3) $x_r(t)x_{\bar{r}}(\bar{t})x_{r+\bar{r}}(v)$, where $r < \bar{r} < r + \bar{r}$, and $v + \bar{v} = N_{r, \bar{r}}t\bar{t}$ ($[e_r, e_{\bar{r}}] = N_{r, \bar{r}}e_{r+\bar{r}}$). These generators arise only in case L is of type A_n , n even. In this case, we assume that 2 is a unit in R . Note that in this case, no generators of type (1) appear [6, p. 877].

3. **Normal subgroups of G_R^1 .** Let $I \not\subseteq H_R$ be an ideal of L_R . Let G_I^1 be the normal closure in G_R^1 of all elements of the forms:

- (4) $x_s(u)$, where $ue_s \in I$, $u = \bar{u}$, and $s = \bar{s}$;
- (5) $x_s(u)x_{\bar{s}}(\bar{u})$, where $ue_s + \bar{u}e_{\bar{s}} \in I$, $s < \bar{s}$, and $s + \bar{s}$ is not a root;
- (6) $x_s(u)x_{\bar{s}}(\bar{u})x_{s+\bar{s}}(w)$, where $ue_s + \bar{u}e_{\bar{s}} \in I$, $w + \bar{w} = N_{s, \bar{s}}u\bar{u}$, and $s < \bar{s} < s + \bar{s}$.

Then, of course, G_I^1 is a normal subgroup of G_R^1 which corresponds to the ideal I of L_R . The natural question that arises now is to what extent distinct ideals of L_R give rise to distinct normal subgroups of G_R^1 . The first observation to be made is [1, 3.4] that if I is an ideal of L_R , then $I \cap E_R = E_J = JE_R$ for some ideal J of R . J is of course uniquely determined by I , but not conversely: $I \cap H_R$ may coincide with or properly contain $H_J = JH_R$. Thus we see that if I and I' are two ideals of L_R which determine the same ideal J of R , then $G_I^1 = G_{I'}^1$, but this sufficient condition is not necessary. A necessary and sufficient condition is given by the following theorem.

THEOREM 1. $G_I^1 = G_{I'}^1$ if and only if $J \cap \bar{J} = J' \cap \bar{J}'$ where $I \cap E_R = JE_R$ and $I' \cap E_R = J'E_R$.

PROOF. Suppose first that $J \cap \bar{J} = J' \cap \bar{J}'$. Consider type (4) generators $x_s(u)$ for G_I^1 where $u = \bar{u} \in J$. Then $\bar{u} \in \bar{J}$, so $u \in J \cap \bar{J}$, hence $u \in J' \cap \bar{J}'$. Thus all such generators also belong to G_I^1 . For type (5) generators $x_s(u)x_{\bar{s}}(\bar{u})$ we have $ue_s + \bar{u}e_{\bar{s}} \in JE_R$, so $ue_s \in JE_R$ and $\bar{u}e_{\bar{s}} \in JE_R$ from [3, 3.4]. Then $u \in J$, $\bar{u} \in \bar{J}$. Since $\bar{u} \in \bar{J}$, we have $\bar{u} \in J \cap \bar{J}$, and since $\bar{u} \in J$, we have $u = \bar{u} \in \bar{J}$, hence $u \in J \cap \bar{J}$, also. Thus both u and \bar{u} are in $J' \cap \bar{J}'$. Thus all these generators belong to G_I^1 . Finally, for type (6) generators $x_s(u)x_{\bar{s}}(\bar{u})x_{s+\bar{s}}(w)$ we again have $ue_s + \bar{u}e_{\bar{s}} \in JE_R$, so u and \bar{u} belong to $J \cap \bar{J} = J' \cap \bar{J}'$. Also then $u\bar{u} \in J \cap \bar{J} = J' \cap \bar{J}'$, hence $w \in J' \cap \bar{J}'$. So all these generators are in G_I^1 . Hence $G_I^1 \subseteq G_I^1$. But interchanging G_I^1 and G_I^1 and performing the same reasoning, we get $G_I^1 \subseteq G_I^1$.

Conversely now, suppose $J \cap \bar{J} \neq J' \cap \bar{J}'$. Then say $y \in J \cap \bar{J}$, $y \notin J' \cap \bar{J}'$. If $y = \bar{y}$ and there are self-conjugate roots s , then we have $x_s(y) \in G_I^1$. But $x_s(y) \notin G_I^1$. For the only way to obtain elements of G_I^1 is as products of conjugates of elements of type (4), (5), and (6), or their inverses [5, p. 53]. Now if u and u' are in $J' \cap \bar{J}'$, then $x_r(u)x_r(u') = x_r(u+u')$, where $u+u' \in J' \cap \bar{J}'$. So $u+u' \neq y$. Also in conjugating any elements of G_I^1 we can conjugate each factor and either lengthen the product, or use the commutator lemma in the form

$$x_r(t)x_s(u)x_r(-t) = x_{r+s}(\pm tu)x_s(-u)$$

[7, p. 24], or use a relation of the form

$$h_r(t)x_s(u)h_r(t)^{-1} = x_s(t^{c(s,r)}u), \text{ or } \omega_r(1)x_s(u)\omega_r(1)^{-1} = x_{wrs}(\pm u),$$

where $h_r(t) = \omega_r(t)\omega_r(-1)$, $\omega_r(t) = x_r(t)x_{-r}(-t^{-1})x_r(t)$, t a unit in R (cf. [6, 5.1 and 7.3]). In the second to last case, observe that if $u \in J' \cap \bar{J}'$ then $t^{c(s,r)}u \in J' \cap \bar{J}'$ also, so $y \neq t^{c(s,r)}u$. In the case involving the commutator lemma, observe that if $u \in J' \cap \bar{J}'$, then $\pm tu \in J' \cap \bar{J}'$, which is an ideal since \bar{J}' is the image of the ideal J' of R under a ring automorphism. Thus no $\pm tu$ will be y . Thus there is no way to obtain $x_s(y) \in G_I^1$. If there are no self-conjugate roots, then we can apply the reasoning just used to $x_s(y)x_{\bar{s}}(\bar{y})$ or $x_s(y)x_{\bar{s}}(\bar{y})x_{s+\bar{s}}(w)$ in G_I^1 , and reach the conclusion that the respective one of these elements is not in G_I^1 , since $ye_s + \bar{y}e_{\bar{s}} \notin I'$. Thus $G_I^1 \neq G_I^1$. This completes the proof.

Now let us define an equivalence relation on the set $\mathcal{I}(R)$ of ideals of R by $J \sim J'$ if and only if $J \cap \bar{J} = J' \cap \bar{J}'$. It is clear that this is an equivalence relation, and Theorem 1 yields the following corollary.

COROLLARY. *The set of normal subgroups G_I^1 defined above is in one-to-one correspondence with the set $\mathcal{I}(R)/\sim$ of equivalence classes of ideals of R .*

What additional information do our results give about the normal structure of G_R^1 ? As one example, we obtain the following result along the lines of Satz 2 of [4].

THEOREM 2. *Suppose G_R^1 has generators of type (1). Then the normal closure N of such an $x_r(t)$ is G_I^1 where $I=JL_R$, $J=\langle t \rangle$, the principal ideal generated by t in R .*

PROOF. Note that there are no type (3) generators by hypothesis. If we use $\omega_r(1)=x_r(1)x_{-r}(-1)x_r(1)$ corresponding to

$$w \in W^1 = \{w \in W \mid w(\bar{s}) = \bar{w}(s)\},$$

where W is the Weyl group, then we can obtain any $x_s(\pm t)$ for $s=\bar{s}$ as a member of N , in view of Corollary 2.8 of [6]. In addition,

$$\begin{aligned} (x_r(t), x_s(1)x_{\bar{s}}(1)) &= (x_r(t), x_s(1))x_s(1)(x_r(t), x_{\bar{s}}(1))x_s(1)^{-1} \\ &= x_{r+s}(\pm t)x_s(1)x_{r+\bar{s}}(\pm t)x_s(1)^{-1} \\ &= x_s(1)x_{r+s}(\pm t)x_{r+\bar{s}}(\pm t)x_s(1)^{-1}, \end{aligned}$$

and hence $x_{r+s}(\pm t)x_{r+\bar{s}}(\pm t)$ is in N . Observe that the plus or minus sign is the same in each factor since it is determined by $N_{r,s}=N_{r,\bar{s}}$ since $(r+s)^- = r+\bar{s}$. Since $r+s \neq (r+s)^- = r+\bar{s}$, we have one type (5) generator in N . But then again using elements corresponding to members of W^1 , we get all such type (5) elements in N . Since all the generators of G_I^1 as a normal subgroup are in N , we have $G_I^1 \subseteq N$, so $G_I^1 = N$, as desired.

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