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THE NORMALITY OF CERTAIN SUBGROUPS OF ELEMENTARY SUBGROUPS OF STEINBERG GROUPS OVER RINGS

JAMES F. HURLEY

ABSTRACT. This paper amends the approach used in an earlier paper to construct, from ideals in the Chevalley algebra L_R over a commutative ring R with identity, normal subgroups of the elementary subgroup G_R^1 of Steinberg's twisted group corresponding to L, a finite dimensional simple Lie algebra over the complex field. The set of normal subgroups so constructed turns out to be in one-to-one correspondence with the set of equivalence classes of ideals of R under an equivalence relation defined in terms of the underlying automorphism of R of order 2.

1. **Introduction.** In [3], to which the reader is referred for all notation not specified here, the same procedure developed in [2] was used with the aim of creating normal subgroups of G_R^1 corresponding to ideals I in the Chevalley algebra L_R of the Lie algebra L over R. R. W. Carter has pointed out to the author that the procedure of [2] fails in general to define a normal subgroup in the context of [3], since the elements $x_r(t)$ which lie in U_R^1 or V_R^1 do not generate the group G_R^1. In the present paper, a modification of the approach found in [3, §4] is used to construct a normal subgroup G_I^1 of G_R^1 corresponding to an ideal I of L_R. The author would like to acknowledge and thank Professor Carter for the suggestion that a condition of the form $ue_s+\bar{u}e_{\bar{s}}\in I$ would be needed in the definition of G_I^1 (see (5) and (6) below).

While such a normal subgroup could be described as the subgroup of G_R^1 generated by a certain set of elements in the manner of [2], these elements are much more complicated than the corresponding generators of G_I in [2] so that such a description becomes quite unwieldy. The normal subgroup G_I^1 in [2] however is the normal closure of the subgroup generated by all elements $x_r(t)$ where $te_r\in I$. It is this fact which serves as the point of departure for the approach taken in the present paper. We define G_I to be the normal closure of the elements of type (4), (5), and

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(6) below. The problem, then, is to determine to what extent distinct ideals of L_R gives rise to distinct normal subgroups of G_R^1. If $I \cap E_R = JE_R$ and $I' \cap E_R = J'E_R$, then we show that $G_I^1 = G_{I'}^1$ if and only if $J \cap \bar{J} = J' \cap \bar{J}'$. Here $E_R = R \otimes_Z E_Z$ where E_Z is the free abelian group on the Chevalley basis elements not in H. We thus obtain a bijective correspondence between the set of normal subgroups G_I^1 and the set of equivalence classes of ideals of R modulo the equivalence relation on the set of ideals of R defined by the condition $J' \cap \bar{J}' = J \cap \bar{J}$.

2. **Generators for G_R^1.** We assume that the Lie algebra L has a symmetry $r \to \bar{r}$ of order 2 of its Coxeter–Dynkin diagram, so that L_R has a semiautomorphism of order 2 [**6**, Lemma 3.2]. We also assume that R has an automorphism $t \to \bar{t}$ of order 2. Then $\sigma: G_R \to G_R$ given by $\sigma(x_r(t)) = x_{\bar{r}}(\bar{t})$ is an automorphism. G_R^1 is the subgroup of G_R generated by U_R^1 and V_R^1, which are the respective intersections of the fixed point group of σ with the maximal unipotent subgroups U_R and V_R generated by the elements $x_r(t)$ where r runs respectively over the positive and negative roots. Then G_R^1 is generated [**6**, Lemma 4.6] by elements of the forms:

(1) $x_r(t)$, where $r = \bar{r}$, $t = \bar{t}$, $r \neq s + \bar{s}$ for any root s;

(2) $x_r(t)x_{\bar{r}}(\bar{t})$, where $r < \bar{r}$, and $r + \bar{r}$ is not a root;

(3) $x_r(t)x_{\bar{r}}(\bar{t})x_{r+\bar{r}}(v)$, where $r < \bar{r} < r + \bar{r}$, and $v + v = N_{r,\bar{r}}t\bar{t}$ ($[e_r, e_{\bar{r}}] = N_{r,\bar{r}}e_{r+\bar{r}}$). These generators arise only in case L is of type A_n, n even. In this case, we assume that 2 is a unit in R. Note that in this case, no generators of type (1) appear [**6**, p. 877].

3. **Normal subgroups of G_R^1.** Let $I \nsubseteq H_R$ be an ideal of L_R. Let G_I^1 be the normal closure in G_R^1 of all elements of the forms:

(4) $x_s(u)$, where $ue_s \in I$, $u = \bar{u}$, and $s = \bar{s}$;

(5) $x_s(u)x_{\bar{s}}(\bar{u})$, where $ue_s + \bar{u}e_{\bar{s}} \in I$, $s < \bar{s}$, and $s + \bar{s}$ is not a root;

(6) $x_s(u)x_{\bar{s}}(\bar{u})x_{s+\bar{s}}(w)$, where $ue_s + \bar{u}e_{\bar{s}} \in I$, $w + w = N_{s,\bar{s}}u\bar{u}$, and $s < \bar{s} < s + \bar{s}$.

Then, of course, G_I^1 is a normal subgroup of G_R^1 which corresponds to the ideal I of L_R. The natural question that arises now is to what extent distinct ideals of L_R give rise to distinct normal subgroups of G_R^1. The first observation to be made is [**1**, 3.4] that if I is an ideal of L_R, then $I \cap E_R = E_J = JE_R$ for some ideal J of R. J is of course uniquely determined by I, but not conversely: $I \cap H_R$ may coincide with or properly contain $H_J = JH_R$. Thus we see that if I and I' are two ideals of L_R which determine the same ideal J of R, then $G_I^1 = G_{I'}^1$, but this sufficient condition is not necessary. A necessary and sufficient condition is given by the following theorem.

THEOREM 1. *$G_I^1 = G_{I'}^1$ if and only if $J \cap \bar{J} = J' \cap \bar{J}'$ where $I \cap E_R = JE_R$ and $I' \cap E_R = J'E_R$.*

PROOF. Suppose first that $J\cap\bar{J}=J'\cap\bar{J}'$. Consider type (4) generators $x_s(u)$ for G^1_I where $u=\bar{u}\in J$. Then $\bar{u}\in\bar{J}$, so $u\in J\cap\bar{J}$, hence $u\in J'\cap\bar{J}'$. Thus all such generators also belong to $G^1_{I'}$. For type (5) generators $x_s(u)x_{\bar{s}}(\bar{u})$ we have $ue_s+\bar{u}e_{\bar{s}}\in JE_R$, so $ue_s\in JE_R$ and $\bar{u}e_{\bar{s}}\in JE_R$ from [3, 3.4]. Then $u\in J$, $\bar{u}\in J$. Since $\bar{u}\in\bar{J}$, we have $\bar{u}\in J\cap\bar{J}$, and since $\bar{u}\in J$, we have $u=\bar{\bar{u}}\in\bar{J}$, hence $u\in J\cap\bar{J}$, also. Thus both u and $\bar{u}$ are in $J'\cap\bar{J}'$. Thus all these generators belong to $G^1_{I'}$. Finally, for type (6) generators $x_s(u)x_{\bar{s}}(\bar{u})x_{s+\bar{s}}(w)$ we again have $ue_s+\bar{u}e_{\bar{s}}\in JE_R$, so u and $\bar{u}$ belong to $J\cap\bar{J}=J'\cap\bar{J}'$. Also then $u\bar{u}\in J\cap\bar{J}=J'\cap\bar{J}'$, hence $w\in J'\cap\bar{J}'$. So all these generators are in $G^1_{I'}$. Hence $G^1_I\subseteq G^1_{I'}$. But interchanging G^1_I and $G^1_{I'}$ and performing the same reasoning, we get $G^1_{I'}\subseteq G^1_I$.

Conversely now, suppose $J\cap\bar{J}\neq J'\cap\bar{J}'$. Then say $y\in J\cap\bar{J}$, $y\notin J'\cap\bar{J}'$. If $y=\bar{y}$ and there are self-conjugate roots s, then we have $x_s(y)\in G^1_I$. But $x_s(y)\notin G^1_{I'}$. For the only way to obtain elements of $G^1_{I'}$ is as products of conjugates of elements of type (4), (5), and (6), or their inverses [5, p. 53]. Now if u and u' are in $J'\cap\bar{J}'$, then $x_r(u)x_r(u')=x_r(u+u')$, where $u+u'\in J'\cap\bar{J}'$. So $u+u'\neq y$. Also in conjugating any elements of $G^1_{I'}$ we can conjugate each factor and either lengthen the product, or use the commutator lemma in the form

$$x_r(t)x_s(u)x_r(-t) = x_{r+s}(\pm tu)x_s(-u)$$

[7, p. 24], or use a relation of the form

$$h_r(t)x_s(u)h_r(t)^{-1} = x_s(t^{c(s,r)}u), \quad \text{or} \quad \omega_r(1)x_s(u)\omega_r(1)^{-1} = x_{w_r s}(\pm u),$$

where $h_r(t)=\omega_r(t)\omega_r(-1)$, $\omega_r(t)=x_r(t)x_{-r}(-t^{-1})x_r(t)$, t a unit in R (cf. [6, 5.1 and 7.3]). In the second to last case, observe that if $u\in J'\cap\bar{J}'$ then $t^{c(s,r)}u\in J'\cap\bar{J}'$ also, so $y\neq t^{c(s,r)}u$. In the case involving the commutator lemma, observe that if $u\in J'\cap\bar{J}'$, then $\pm tu\in J'\cap\bar{J}'$, which is an ideal since $\bar{J}'$ is the image of the ideal J' of R under a ring automorphism. Thus no $\pm tu$ will be y. Thus there is no way to obtain $x_s(y)\in G^1_{I'}$. If there are no self-conjugate roots, then we can apply the reasoning just used to $x_s(y)x_{\bar{s}}(\bar{y})$ or $x_s(y)x_{\bar{s}}(\bar{y})x_{s+\bar{s}}(w)$ in G^1_I, and reach the conclusion that the respective one of these elements is not in $G^1_{I'}$, since $ye_s+\bar{y}e_{\bar{s}}\notin I'$. Thus $G^1_{I'}\neq G^1_I$. This completes the proof.

Now let us define an equivalence relation on the set $\mathscr{I}(R)$ of ideals of R by $J\sim J'$ if and only if $J\cap\bar{J}=J'\cap\bar{J}'$. It is clear that this is an equivalence relation, and Theorem 1 yields the following corollary.

COROLLARY. *The set of normal subgroups G^1_I defined above is in one-to-one correspondence with the set $\mathscr{I}(R)/\sim$ of equivalence classes of ideals of R.*

What additional information do our results give about the normal structure of G^1_R? As one example, we obtain the following result along the lines of Satz 2 of [4].

THEOREM 2. *Suppose G_R^1 has generators of type* (1). *Then the normal closure N of such an $x_r(t)$ is G_I^1 where $I=JL_R$, $J=\langle t\rangle$, the principal ideal generated by t in R.*

PROOF. Note that there are no type (3) generators by hypothesis. If we use $\omega_r(1)=x_r(1)x_{-r}(-1)x_r(1)$ corresponding to

$$w \in W^1 = \{w \in W \mid w(\bar{s}) = \bar{w}(s)\},$$

where W is the Weyl group, then we can obtain any $x_s(\pm t)$ for $s=\bar{s}$ as a member of N, in view of Corollary 2.8 of [**6**]. In addition,

$$\begin{aligned}(x_r(t), x_s(1)x_{\bar{s}}(1)) &= (x_r(t), x_s(1))x_s(1)(x_r(t), x_{\bar{s}}(1))x_s(1)^{-1}\\ &= x_{r+s}(\pm t)x_s(1)x_{r+\bar{s}}(\pm t)x_s(1)^{-1}\\ &= x_s(1)x_{r+s}(\pm t)x_{r+\bar{s}}(\pm t)x_s(1)^{-1},\end{aligned}$$

and hence $x_{r+s}(\pm t)x_{r+\bar{s}}(\pm t)$ is in N. Observe that the plus or minus sign is the same in each factor since it is determined by $N_{r,s}=N_{r,\bar{s}}$ since $\overline{(r+s)}=r+\bar{s}$. Since $r+s\neq\overline{(r+s)}=r+\bar{s}$, we have one type (5) generator in N. But then again using elements corresponding to members of W^1, we get all such type (5) elements in N. Since all the generators of G_I^1 as a normal subgroup are in N, we have $G_I^1\subseteq N$, so $G_I^1=N$, as desired.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, CONNECTICUT 06268