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A COUNTEREXAMPLE IN THE CLASSIFICATION OF OPEN RIEMANN SURFACES

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ABSTRACT. An HD-function (harmonic and Dirichlet-finite) ω on a Riemann surface R is called HD-minimal if $\omega>0$ and every HD-function ω' with $0\leqq\omega'\leqq\omega$ reduces to a constant multiple of ω. An $HD^{\sim}$-function is the limit of a decreasing sequence of positive HD-functions and $HD^{\sim}$-minimality is defined as in HD-functions. The purpose of the present note is to answer in the affirmative the open question: Does there exist a Riemann surface which carries an $HD^{\sim}$-minimal function but no HD-minimal functions?

An HD-function (harmonic and Dirichlet-finite) ω on a Riemann surface R is called HD-minimal if ω is positive and every HD-function ω' with $0<\omega'\leqq\omega$ reduces to a constant multiple of ω on R. Let $\{\omega_n\}$ be a decreasing sequence of positive HD-functions on R. Then its limit is harmonic on R, and called an $HD^{\sim}$-function on R. $HD^{\sim}$-minimality can be defined as for HD-minimal functions. Denote by U_{HD} (resp. $U_{HD^{\sim}}$) the class of open Riemann surfaces on which an HD-minimal (resp. $HD^{\sim}$-minimal) function exists (Constantinescu and Cornea [2]).

It is well known (Nakai [5], see also Sario-Nakai [7, p. 186]) that the inclusion $U_{HD}\subset U_{HD^{\sim}}$ holds. The purpose of the present paper is to show that the inclusion is strict. For Riemannian manifolds of $\dim\geqq 3$ its strictness was established in Kwon [4]. For the sake of completeness we shall also give a somewhat simplified proof.

It should be noted that our reasoning is suggested by ingenious examples of Toki ([8], [9]); see also Sario [6]. The author is very grateful to the referee for his helpful suggestions.

1. First we demonstrate a hyperbolic Riemann surface which does not carry nonconstant positive harmonic functions (Toki [9]). For the sake of simplicity we follow the construction and the notation in Ahlfors and Sario [1, pp. 256–261].

Our surface will be obtained from the unit disk $U:|z|<1$ by identifying,

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pairwise or cyclically, edges of infinitely many radial slits. For a slit $S=\{re^{i\alpha}|0<a\leqq r\leqq b<1\}$, set $S^+=\{re^{i(\alpha+0)}|a\leqq r\leqq b\}$ and $S^-=\{re^{i(\alpha-0)}|a\leqq r\leqq b\}$. Two radial slits S_1 and S_2 are identified pairwise if S_1^+ is connected with S_2^- and S_1^- with S_2^+. The radial slits $S_1, S_2, \cdots, S_n$ are identified cyclically if S_1^+ is connected with S_2^-, S_2^+ with S_3^-, etc., and finally S_n^+ with S_1^-. Here we understand that all the slits extend between two concentric circles.

For a pair (h, k) of positive integers h, k, set $\mu=(2h-1)2^{k-1}$. With each μ we associate $2^{k+5\mu}$ radial slits, equally spaced and one being on the positive real axis, such that their end points lie on $|z|=r_{4\mu-2}$ and $|z|=r_{4\mu-1}$, where $\log r_\mu=-2^{-\mu}$ for all $\mu\geqq 1$. A slit associated with $\mu=\mu(h, k)$ will be called of rank μ and type k. For each $k\geqq 1$ denote by S_{ik} the sectors: $2\pi i\cdot 2^{-k}\leqq \theta\leqq 2\pi(i+1)\cdot 2^{-k}$, $0\leqq i<2^k$. The slits of type k on the rays $\theta=2\pi i\cdot 2^{-k}$ will be identified cyclically. The remaining slits of the same type are identified pairwise within each sector S_{ik}, symmetrically with respect to its bisecting ray. Let $\tilde{U}$ be the resulting Riemann surface.

LEMMA 1. *The Riemann surface $\tilde{U}$ is hyperbolic, but every positive harmonic function on $\tilde{U}$ reduces to a constant.*

For a proof we refer the reader to Ahlfors and Sario [**1**, pp. 256–261].

2. Denote by U_0 the Riemann surface obtained from $\tilde{U}$ by deleting all the radial slits

$$\sum_{hk}^{\nu} = \{re^{i\theta}\mid -2^{-4\mu}\leqq \log r\leqq -2^{-4\mu-1}, \theta=2\pi\nu\cdot 2^{-4\mu}\}$$

for $1\leqq\nu\leqq 2^{4\mu}$. Let $\{U_0(l)\}_1^\infty$ be a sequence of duplicates of U_0. For each fixed $k\geqq 1$ and subsequently for $j\geqq 0$ and $1\leqq l\leqq 2^{k-1}$, join $U_0(l+2^kj)$, crosswise along all the slits $\sum_{hk}^{\nu}\ (h\geqq 1)$, with $U_0(l+2^{k-1}+2^kj)$ (cf. Sario [**6**]). The resulting Riemann surface R is an infinitely sheeted covering surface of $\tilde{U}$. Let $\pi: R\to\tilde{U}$ be the natural projection.

LEMMA 2. *The Riemann surface R carries no nonconstant bounded harmonic functions. Furthermore every bounded harmonic function u on the subregion*

$$G=\{x\in R\mid |\pi(x)|>r_1\}$$

takes the same value on $\pi^{-1}(z)$ for each $z\in\tilde{U}$ whenever it continuously vanishes on the relative boundary

$$\partial G=\{x\in R\mid |\pi(x)|=r_1\},$$

where $\log r_1=-2^{-1}$.

For the proof the reader is referred to Sario-Nakai [**7**, pp. 178–181].

3. For each integer $l \geqq 1$, consider the subset of R:

$$R_l = \left[\bigcup_{j=1}^{l-1} G_j\right] \cup \left[\bigcup_{j=l}^{\infty} U_0(j)\right]$$

where $G_j = \{x \in U_0(j) \mid |\pi(x)| > r_1\}$. It is obvious that $G = \bigcup_{j=1}^{\infty} G_j$ and the Riemann surface G is an infinitely sheeted covering surface of the "annulus" $\{z \in \tilde{U} \mid |z| > r_1\}$.

We are now ready to state our main result (cf. Kwon [4]):

THEOREM 1. *The Riemann surface G carries a unique (up to constant factors) $HD^{\sim}$-minimal function but no HD-minimal functions. Thus the inclusion $U_{HD} \subset U_{HD^{\sim}}$ is strict for Riemann surfaces.*

The proof will be given in §§4–5. For convenience we shall follow the notation and terminology in Sario-Nakai [7]. All results needed concerning the Royden and Wiener compactifications can be found in Sario-Nakai [7, Chapters 3 and 4].

4. For each $m \geqq 1$ choose $u_m \in HBD(R_m)$, the class of bounded Dirichlet-finite harmonic functions on R_m, such that $0 \leqq u_m \leqq 1$ on R, $u_m \equiv 0$ on $\bigcup_{j=1}^{m-1} [U_0(j) - G_j]$, and $u_m \equiv 1$ on the Royden harmonic boundary of R. In view of the fact that R is hyperbolic and carries no nonconstant bounded harmonic functions, the Wiener harmonic boundary Δ_N and the Royden harmonic boundary Δ_M of R consist of single points. Therefore the maximum principle yields

$$u_m(x) \geqq 1 - (\log|\pi(x)|)/\log r_1$$

on G. Clearly $u_m \geqq u_{m+1}$ and therefore the sequence $\{u_m\}$ converges, uniformly on compact subsets of G, to an $HD^{\sim}$-function u on G. It is obvious that $0 < u < 1$ on G and $u \equiv 0$ on $R - G$.

We claim that the function u is $HD^{\sim}$-minimal on G. In fact let $v \in HD^{\sim}(G)$, the class of $HD^{\sim}$-functions on G, satisfy $0 < v \leqq u$ on G. Since

$$0 \leqq \limsup_{x \in G, x \to y} v(x) \leqq \limsup_{x \in G, x \to y} u(x) = 0$$

for every $y \in \partial G$, the function v can also be continuously extendable to R with $v | R - G \equiv 0$. Again by the maximum principle we have $v = \alpha u$ on G, where $\alpha = \lim_{x \to \Delta_N} v(x)$ the limit being taken in the Wiener compactification of R.

5. Suppose that the function u is HD-minimal on G. Then u must have a finite Dirichlet integral over G. But u has a continuous extension to $G \cup \partial G$ with $u | \partial G \equiv 0$. Therefore u must attain the same value at all the points in G which lie over the same point in $\tilde{U}$, a contradiction.

Finally it remains to show that every $HD^{\sim}$-minimal function on G is a constant multiple of u. Let ω be another $HD^{\sim}$-minimal function on G. Choose a point $q \in \Delta_{M,G}$, the Royden harmonic boundary of G, such that q has a positive harmonic measure and $\lim\sup_{x\in G, x\to q'} \omega(x)=0$ for almost all $q' \in \Delta_{M,G}-\{q\}$ relative to a harmonic measure μ for G. Then ω has an integral representation in the form:

$$\omega(x) = \int_{\Delta_{M,G}} P(x, y)\bar{\omega}(y)\, d\mu(y)$$

on G, where $P(x, y)$ is the harmonic kernel and $\bar{\omega}(y)=\lim\sup_{x\in G, x\to y} \omega(y)$ for $y \in \Delta_{M,G}$ (Nakai [5]; see also Sario-Nakai [7, p. 183]).

Let $j: G^*\to \bar{G} \subset R^*$ be the subjective continuous mapping such that $j(x)=x$ on G and $f(x)=f(j(x))$ for all $x \in G^*$, the Royden compactification of G, and $f \in M(R)$, the Royden algebra of R. Here $\bar{G}$ is the closure of G in the Royden compactification R^* of R. Note that a Borel subset $E \subset \partial G$ has a positive harmonic measure if and only if $j^{-1}(E)$ has a positive harmonic measure (cf. Sario-Nakai [7, p. 192]). Therefore $j(q) \notin \partial G$. In view of Lemma 2 it is obvious that $j(q) \in \mathrm{Cl}(\partial G)$, the closure being taken in R^*.

For each $m \geqq 1$, $u_m(q)=u_m(j(q))=1$ since $j(q) \in \mathrm{Cl}(\partial G)-\partial G$. Thus by virtue of integral representations of ω and u_m, it is not difficult to see that $0<\omega \leqq \beta u_m$ on G, where $\beta=\bar{\omega}(q)$. Therefore $0<\omega\leqq\beta u$ on G and ω is a constant multiple of u on G as in §4.

This completes the proof of Theorem 1.

6. We turn to Riemannian n-manifolds for $n\geqq 3$. Our manifold will be a submanifold of an infinitely sheeted covering manifold of the n-dimensional Euclidean space R^n. Note that R^n and $\tilde{U}$ share the properties stated in Lemma 1.

For the construction replace the radial slits $\sum_{hk}^{\nu}$ $(1\leqq \nu \leqq 2^{4\mu})$ by the hemispheres

$$H_{hk} = \{8^{\mu} x \in R^n \mid |x| = 1 \text{ and } x^1 \geqq 0\}$$

where $8^{\mu}x=(8^{\mu}x^1, \cdots, 8^{\mu}x^n)$ for $x=(x^1, \cdots, x^n)$. Denote by M the infinitely sheeted covering manifold of R^n, constructed exactly in the same way as in R. The counterparts for Lemma 2 and Theorem 1 now read:

LEMMA 3. *The Riemannian n-manifold M carries no nonconstant bounded harmonic functions. Every bounded harmonic function on the submanifold*

$$N = \{x \in M \mid |\pi(x)| > 1\}$$

attains the same value at all the points in M which lie over the same point *in R^n if it continuously vanishes on*

$$\partial N = \{x \in M \mid |\pi(x)| = 1\}.$$

THEOREM 2. *The Riemannian n-manifold N ($n \geqq 3$) carries a unique (up to constant factors) $HD^{\sim}$-minimal function but no HD-minimal functions.*

The proofs of Lemma 3 and Theorem 2 are similar to those of Lemma 2 and Theorem 1 (cf. Kwon [3]).

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