A COUNTEREXAMPLE IN THE CLASSIFICATION OF OPEN RIEMANN SURFACES

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ABSTRACT. An $HD$-function (harmonic and Dirichlet-finite) $\omega$ on a Riemann surface $R$ is called $HD$-minimal if $\omega>0$ and every $HD$-function $\omega'$ with $0 \leq \omega' \leq \omega$ reduces to a constant multiple of $\omega$. An $HD^*$-function is the limit of a decreasing sequence of positive $HD$-functions and $HD^*$-minimality is defined as in $HD$-functions. The purpose of the present note is to answer in the affirmative the open question: Does there exist a Riemann surface which carries an $HD^*$-minimal function but no $HD$-minimal functions?

An $HD$-function (harmonic and Dirichlet-finite) $\omega$ on a Riemann surface $R$ is called $HD$-minimal if $\omega$ is positive and every $HD$-function $\omega'$ with $0 < \omega' \leq \omega$ reduces to a constant multiple of $\omega$ on $R$. Let $\{\omega_n\}$ be a decreasing sequence of positive $HD$-functions on $R$. Then its limit is harmonic on $R$, and called an $HD^*$-function on $R$. $HD^*$-minimality can be defined as for $HD$-minimal functions. Denote by $U_{HD}$ (resp. $U_{HD^*}$) the class of open Riemann surfaces on which an $HD$-minimal (resp. $HD^*$-minimal) function exists (Constantinescu and Cornea [2]).

It is well known (Nakai [5], see also Sario-Nakai [7, p. 186]) that the inclusion $U_{HD} \subset U_{HD^*}$ holds. The purpose of the present paper is to show that the inclusion is strict. For Riemannian manifolds of dim $\geq 3$ its strictness was established in Kwon [4]. For the sake of completeness we shall also give a somewhat simplified proof.

It should be noted that our reasoning is suggested by ingenious examples of Toki ([8], [9]); see also Sario [6]. The author is very grateful to the referee for his helpful suggestions.

1. First we demonstrate a hyperbolic Riemann surface which does not carry nonconstant positive harmonic functions (Toki [9]). For the sake of simplicity we follow the construction and the notation in Ahlfors and Sario [1, pp. 256-261].

Our surface will be obtained from the unit disk $U$: $|z|<1$ by identifying,
pairwise or cyclically, edges of infinitely many radial slits. For a slit 
\( S = \{ re^{i\theta} | 0 < a \leq r \leq b < 1 \} \), set 
\( S^+ = \{ re^{i(\theta + \pi/2)} | a \leq r \leq b \} \) and 
\( S^- = \{ re^{i(\theta - \pi/2)} | a \leq r \leq b \} \). Two radial slits \( S_1 \) and \( S_2 \) are identified pairwise if \( S_1^+ \) is connected 
with \( S_2^- \) and \( S_1^- \) with \( S_2^+ \). The radial slits \( S_1, S_2, \cdots, S_n \) are identified 
cyclically if \( S_1^+ \) is connected with \( S_2^- \), \( S_2^+ \) with \( S_3^- \), etc., and finally \( S_n^+ \) with 
\( S_1^- \). Here we understand that all the slits extend between two concentric 
circles.

For a pair \((h, k)\) of positive integers \( h, k \), set \( \mu = (2h - 1)2^{k-1} \). With each \( \mu \) 
we associate \( 2^{k+5h} \) radial slits, equally spaced and one being on the positive 
real axis, such that their end points lie on \( |z| = r_{4h-2} \) and \( |z| = r_{4h-1} \), where 
\( \log r_{\mu} = -2^{-\mu} \) for all \( \mu \geq 1 \). A slit associated with \( \mu = \mu(h, k) \) will be called 
of rank \( \mu \) and type \( k \). For each \( k \geq 1 \) denote by \( S_{ik} \) the sectors: 
\( 2\pi i \cdot 2^{-k} \leq \theta \leq 2\pi (i+1) \cdot 2^{-k}, 0 \leq i < 2^k \). The slits of type \( k \) on the rays \( \theta = 2\pi i \cdot 2^{-k} \) 
will be identified cyclically. The remaining slits of the same type are 
identified pairwise within each sector \( S_{ik} \), symmetrically with respect to 
its bisecting ray. Let \( \bar{U} \) be the resulting Riemann surface.

**Lemma 1.** The Riemann surface \( \bar{U} \) is hyperbolic, but every positive harmonic function on \( \bar{U} \) reduces to a constant.

For a proof we refer the reader to Ahlfors and Sario [1, pp. 256–261].

2. Denote by \( U_0 \) the Riemann surface obtained from \( \bar{U} \) by deleting 
all the radial slits

\[
\sum_{hk} = \{ re^{i\theta} | -2^{-4h} \leq \log r \leq -2^{-4h-1}, \theta = 2\pi v \cdot 2^{4h} \}
\]

for \( 1 \leq v \leq 2^{4h} \). Let \( \{ U_0(l) \}_{l=1}^{2^{4h}} \) be a sequence of duplicates of \( U_0 \). For each 
fixed \( k \geq 1 \) and subsequently for \( j \geq 0 \) and \( 1 \leq l \leq 2^{k-1} \), join \( U_0(l+2^j) \), 
crosswise along all the slits \( \sum_k (h \geq 1) \), with \( U_0(l+2^{k-1}+2^j) \) (cf. Sario 
[6]). The resulting Riemann surface \( R \) is an infinitely sheeted covering 
surface of \( \bar{U} \). Let \( \pi: R \rightarrow \bar{U} \) be the natural projection.

**Lemma 2.** The Riemann surface \( R \) carries no nonconstant bounded harmonic functions. Furthermore every bounded harmonic function \( u \) on the subregion

\[
G = \{ x \in R | |\pi(x)| > r_1 \}
\]

takes the same value on \( \pi^{-1}(z) \) for each \( z \in \bar{U} \) whenever it continuously 
vanishes on the relative boundary

\[
\partial G = \{ x \in R | |\pi(x)| = r_1 \},
\]

where \( \log r_1 = -2^{-1} \).

For the proof the reader is referred to Sario-Nakai [7, pp. 178–181].
3. For each integer \( l \geq 1 \), consider the subset of \( R \):
\[
R_l = \left[ \bigcup_{j=1}^{l-1} G_j \right] \cup \left[ \bigcup_{j=1}^\infty \mathcal{U}_0(j) \right]
\]
where \( G_j = \{ x \in \mathcal{U}_0(j) : |\pi(x)| > r_j \} \). It is obvious that \( G = \bigcup_{j=1}^\infty G_j \) and the Riemann surface \( G \) is an infinitely sheeted covering surface of the "annulus" \( \{ z \in \bar{U} : |z| > r_j \} \).

We are now ready to state our main result (cf. Kwon [4]):

**Theorem 1.** The Riemann surface \( G \) carries a unique (up to constant factors) \( HD^\infty \)-minimal function but no \( HD \)-minimal functions. Thus the inclusion \( U_{HD} \subset U_{HD^\infty} \) is strict for Riemann surfaces.

The proof will be given in §§4–5. For convenience we shall follow the notation and terminology in Sario-Nakai [7]. All results needed concerning the Royden and Wiener compactifications can be found in Sario-Nakai [7, Chapters 3 and 4].

4. For each \( m \geq 1 \) choose \( u_m \in HBD(R_m) \), the class of bounded Dirichlet-finite harmonic functions on \( R_m \), such that \( 0 \leq u_m \leq 1 \) on \( R \), \( u_m \equiv 0 \) on \( \bigcup_{j=1}^{m-1} [\mathcal{U}_0(j) - G_j] \), and \( u_m \equiv 1 \) on the Royden harmonic boundary of \( R \). In view of the fact that \( R \) is hyperbolic and carries no nonconstant bounded harmonic functions, the Wiener harmonic boundary \( \Delta_W \) and the Royden harmonic boundary \( \Delta_M \) of \( R \) consist of single points. Therefore the maximum principle yields
\[
u_m(x) \geq 1 - (\log |\pi(x)|) / \log r_1
\]
on \( G \). Clearly \( u_m \geq u_{m+1} \) and therefore the sequence \( \{u_m\} \) converges, uniformly on compact subsets of \( G \), to an \( HD^\infty \)-function \( u \) on \( G \). It is obvious that \( 0 < u < 1 \) on \( G \) and \( u \equiv 0 \) on \( R - G \).

We claim that the function \( u \) is \( HD^\infty \)-minimal on \( G \). In fact let \( v \in HD^\infty(G) \), the class of \( HD^\infty \)-functions on \( G \), satisfy \( 0 < v \leq u \) on \( G \). Since
\[
0 \leq \limsup_{x \to y} v(x) \leq \limsup_{x \to y} u(x) = 0
\]
for every \( y \in \partial G \), the function \( v \) can also be continuously extendable to \( R \) with \( v|R - G \equiv 0 \). Again by the maximum principle we have \( v = xu \) on \( G \), where \( x = \lim_{x \to \Delta_M} v(x) \) the limit being taken in the Wiener compactification of \( R \).

5. Suppose that the function \( u \) is \( HD \)-minimal on \( G \). Then \( u \) must have a finite Dirichlet integral over \( G \). But \( u \) has a continuous extension to \( G \cup \partial G \) with \( u|\partial G \equiv 0 \). Therefore \( u \) must attain the same value at all the points in \( G \) which lie over the same point in \( \bar{U} \), a contradiction.
Finally it remains to show that every $H^D$-minimal function on $G$ is a constant multiple of $u$. Let $\omega$ be another $H^D$-minimal function on $G$. Choose a point $q \in \Delta_{M, G}$, the Royden harmonic boundary of $G$, such that $q$ has a positive harmonic measure and $\limsup_{p \to q} \omega(x) = 0$ for almost all $q' \in \Delta_{M, G} - \{q\}$ relative to a harmonic measure $\mu$ for $G$. Then $\omega$ has an integral representation in the form:

$$\omega(x) = \int_{\Delta_{M, G}} P(x, y)\bar{\omega}(y)\,d\mu(y)$$

on $G$, where $P(x, y)$ is the harmonic kernel and $\bar{\omega}(y) = \limsup_{p \to q} \omega(y)$ for $y \in \Delta_{M, G}$ (Nakai [5]; see also Sario-Nakai [7, p. 183]).

Let $j : G^* \to \hat{G} \subset R^*$ be the subjective continuous mapping such that $j(x) = x$ on $G$ and $f(x) = f(j(x))$ for all $x \in G^*$, the Royden compactification of $G$, and $f \in M(R)$, the Royden algebra of $R$. Here $\hat{G}$ is the closure of $G$ in the Royden compactification $R^*$ of $R$. Note that a Borel subset $E \subset \partial G$ has a positive harmonic measure if and only if $j^{-1}(E)$ has a positive harmonic measure (cf. Sario-Nakai [7, p. 192]). Therefore $j(q) \notin \partial G$.

In view of Lemma 2 it is obvious that $j(q) \in \text{Cl}(\partial G)$, the closure being taken in $R^*$.

For each $m \geq 1$, $u_m(q) = u_m(j(q)) = 1$ since $j(q) \in \text{Cl}(\partial G) - \partial G$. Thus by virtue of integral representations of $\omega$ and $u_m$, it is not difficult to see that $0 < \omega \leq \beta u_m$ on $G$, where $\beta = \bar{\omega}(q)$. Therefore $0 < \omega \leq \beta u$ on $G$ and $\omega$ is a constant multiple of $u$ on $G$ as in §4.

This completes the proof of Theorem 1.

6. We turn to Riemannian $n$-manifolds for $n \geq 3$. Our manifold will be a submanifold of an infinitely sheeted covering manifold of the $n$-dimensional Euclidean space $R^n$. Note that $R^n$ and $\hat{U}$ share the properties stated in Lemma 1.

For the construction replace the radial slits $\sum^\infty_{v=1} (1 \leq v \leq 2^{2m})$ by the hemispheres

$$H_{hk} = \{8^v x \in R^n \mid |x| = 1 \text{ and } x^1 \geq 0\}$$

where $8^v x = (8^v x^1, \cdots, 8^v x^n)$ for $x = (x^1, \cdots, x^n)$. Denote by $M$ the infinitely sheeted covering manifold of $R^n$, constructed exactly in the same way as in $R$. The counterparts for Lemma 2 and Theorem 1 now read:

**Lemma 3.** The Riemannian $n$-manifold $M$ carries no nonconstant bounded harmonic functions. Every bounded harmonic function on the submanifold

$$N = \{x \in M \mid |\pi(x)| > 1\}$$

attains the same value at all the points in $M$ which lie over the same point.
in $\mathbb{R}^n$ if it continuously vanishes on
$$\partial N = \{ x \in M \mid |\pi(x)| = 1 \}. $$

**Theorem 2.** The Riemannian $n$-manifold $N$ ($n \geq 3$) carries a unique (up to constant factors) $HD^-$-minimal function but no $HD$-minimal functions.

The proofs of Lemma 3 and Theorem 2 are similar to those of Lemma 2 and Theorem 1 (cf. Kwon [3]).

**References**


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