PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 2, February 1974

AN INTERPOLATION PROBLEM FOR COEFFICIENTS OF H^∞ FUNCTIONS[1]

JOHN J. F. FOURNIER

ABSTRACT. H^∞ denotes the space of all bounded functions g on the unit circle whose Fourier coefficients $\hat{g}(n)$ are zero for all negative n. It is known that, if $\{n_k\}_{k=0}^\infty$ is a sequence of nonnegative integers with $n_{k+1}>(1+\delta)n_k$ for all k, and if $\sum_{k=0}^\infty |v_k|^2<\infty$, then there is a function g in H^∞ with $\hat{g}(n_k)=v_k$ for all k. Previous proofs of this fact have not indicated how to construct such H^∞ functions. This paper contains a simple, direct construction of such functions. The construction depends on properties of some polynomials similar to those introduced by Shapiro and Rudin. There is also a connection with a type of Riesz product studied by Salem and Zygmund.

We shall present the construction in §1 and discuss its relation to other results in §2. First we recall some notation.

For $p<\infty$, L^p denotes the space of all measurable functions on the unit circle whose p'th power is integrable; as usual, functions are identified if they agree almost everywhere. L^∞ denotes the space of all essentially bounded measurable functions on the unit circle; for g in L^∞, $\|g\|_\infty$ denotes the essential supremum of $|g|$. The Fourier coefficients of an L^1 function g are defined by

$$\hat{g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{i\theta})e^{-in\theta}\,d\theta \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

Finally, H^∞ is taken to be the space of all g in L^∞ which have $\hat{g}(n)=0$ for all $n<0$. It can be shown [4, p. 39] or [5, Theorem 3.12, p. 88] that this version of H^∞ is isomorphic to the space of all bounded analytic functions in the open unit disc, but this fact will not be used here.

1. The construction depends on the fact that, for complex numbers a, b, and v,

$$(1) \qquad |a + vb|^2 + |b - \bar{v}a|^2 = (1 + |v|^2)(|a|^2 + |b|^2).$$

Received by the editors March 13, 1972.

AMS (MOS) subject classifications (1970). Primary 30A10, 30A78, 42A44; Secondary 26A82, 30A80, 42A04, 42A16, 42A80.

Key words and phrases. Interpolation by Fourier coefficients, H^∞ function, Rudin-Shapiro polynomials, Hadamard set, $\Lambda(2)$ set, Riesz product.

[1] Research partially supported by National Research Council of Canada operating grant number A-4822.

This identity plays the same role here as the parallelogram law does in the method of Shapiro and Rudin [**10**, p. 35] and [**7**, pp. 855–856].

THEOREM. *Let $\{n_k\}_{k=0}^\infty$ be a sequence of nonnegative integers so that, for some $\delta>0$, $n_{k+1}>(1+\delta)n_k$ for all k. Let $\sum_{k=0}^\infty |v_k|^2<\infty$. Then there is a function g in H^∞ with $\hat{g}(n_k)=v_k$ for all k.*

PROOF. We begin with the basic case, when $\delta\geqq 1$. Let $z=e^{i\theta}$. Define trigonometric polynomials g_k and h_k by:

$$g_0(z) = v_0 z^{n_0}, \qquad h_0(z) \equiv 1,$$

and, for $k>0$,

$$g_k(z) = g_{k-1}(z) + v_k z^{n_k} h_k(z)$$

and

$$h_k(z) = h_{k-1}(z) - \bar{v}_k z^{-n_k} g_k(z).$$

We claim that the polynomials g_k have the following properties:

(a) $$\|g_k\|_\infty^2 \leqq \prod_{j=0}^{k} (1 + |v_j|^2).$$

(b) $$g_k \in H^\infty.$$

(c) $$\hat{g}_k(n_j) = v_j \quad \text{for all } j \leqq k.$$

Suppose for a moment that these assertions hold. Because $\sum_{j=0}^\infty |v_j|^2<\infty$, the product $\prod_{j=0}^\infty (1+|v_j|^2)$ converges and the sequence $\{\|g_k\|_\infty\}_{k=0}^\infty$ is bounded. Therefore the sequence $\{g_k\}_{k=0}^\infty$ has a weak star limit point, g say, in L^∞. Since $\hat{g}(n)$ is a limit point of $\{\hat{g}_k(n)\}_{k=0}^\infty$ we have, by (b), that g is in H^∞ and, by (c), that $\hat{g}(n_j)=v_j$ for all j. Thus g has the desired properties.

To verify assertion (a) we note that, by (1),

$$|g_k(z)|^2 + |h_k(z)|^2 = (1 + |v_k|^2) \cdot (|g_{k-1}(z)|^2 + |h_{k-1}(z)|^2).$$

As $|g_0(z)|^2+|h_0(z)|^2=1+|v_0|^2$, we have by induction on k that

$$|g_k(z)|^2 + |h_k(z)|^2 = \prod_{j=0}^{k} (1 + |v_j|^2),$$

and (a) follows.

Now let G_k denote the support of $\hat{g}_k$ and H_k denote the support of $\hat{h}_k$. We shall prove by induction on k that

(2) $$G_k \subset [0, n_k]$$

and

(3) $$H_k \subset [-n_k, 0].$$

These assertions clearly hold when $k=0$. Supposing that (2) and (3) hold for some index k, consider the formula

$$g_{k+1}(z) = g_k(z) + v_{k+1}z^{n_{k+1}}h_k(z) = g_k(z) + f_{k+1}(z), \quad \text{say.}$$

By (3), the support of $\hat{f}_{k+1}$ is contained in the interval $[n_{k+1}-n_k, n_{k+1}]$. Hence

$$(4) \qquad G_{k+1} \subset G_k \cup [n_{k+1} - n_k, n_{k+1}] \subset [0, n_{k+1}], \quad \text{as required.}$$

Similarly $H_{k+1} \subset [-n_{k+1}, 0]$. Thus assertions (2) and (3) hold for all k. It follows from (2) that $g_k \in H^\infty$; so (b) holds.

Next, because $n_{k+1} > 2n_k$, the intervals $[0, n_k]$ and $[n_{k+1}-n_k, n_{k+1}]$ are disjoint. Therefore $\hat{g}_k$ and $\hat{f}_{k+1}$ have disjoint supports and

$$(5) \qquad \hat{g}_{k+1}(n) = \hat{g}_k(n) \quad \text{for all } n \leqq n_k.$$

Similarly $\hat{h}_{k+1}(n) = \hat{h}_k(n)$ for all $n \geqq -n_k$. It follows that $\hat{h}_k(0)=1$ for all k. By (2), $\hat{g}_k(n_{k+1})=0$; hence

$$(6) \qquad \hat{g}_{k+1}(n_{k+1}) = \hat{f}_{k+1}(n_{k+1}) = v_{k+1}\hat{h}_k(0) = v_{k+1}.$$

Combining (5) and (6) we see that $\hat{g}_k(n_j)=v_j$ for all $j \leqq k$; so (c) holds. This completes the proof for the case when $\delta \geqq 1$.

Before taking up the case when $\delta < 1$ we mention two more facts about the polynomials g_k. First, it follows from (5) that the weak star limit g discussed above is unique and that g_k is the partial sum of order n_k of the Fourier series of g. Indeed, since the subsequence of partial sums $\{g_k\}_{k=1}^{\infty}$ is uniformly bounded, it follows, without appeal to weak star compactness, that the corresponding trigonometric series must be the Fourier series of a bounded function [**12**, vol. I, p. 148].

Second, it is clear from (4) that $G_k \subset \{n_0\} \cup [n_1-n_0, n_1] \cup \cdots \cup [n_k-n_{k-1}, n_k]$. Therefore, the support of $\hat{g}$ is contained in $\{n_0\} \cup [n_1-n_0, n_1] \cup \cdots \cup [n_k-n_{k-1}, n_k] \cup \cdots$.

When $\delta < 1$, we must split $\{n_k\}$ into a finite number of subsequences, so that the above method works for each subsequence, and so that the functions g corresponding to different subsequences have disjointly supported transforms. The procedure is like the one for Riesz products [**5**, pp. 108–109] or [**12**, vol. II, p. 131].

Let q be the smallest integer such that $\delta(1+\delta)^{q-1} \geqq 1$. Split $\{n_k\}$ and $\{v_k\}$ into q subsequences in each of which the index k runs through an arithmetic progression with period q. Let $\{n_{k_i}\}_{i=0}^{\infty}$ be such a subsequence. Since $k_{i+1}-k_i=q$, we have

$$n_{k_{i+1}} > (1+\delta)^q n_{k_i} = \delta(1+\delta)^{q-1}\frac{(1+\delta)}{\delta}\, n_{k_i} > 2n_{k_i},$$

by the choice of q. So the method used above applied to this subsequence yields a function g in H^∞ with $\hat{g}(n_{k_i})=v_{k_i}$ for all i. Moreover $\hat{g}$ is supported by $\{n_{k_0}\}\cup[n_{k_1}-n_{k_0}, n_{k_1}]\cup\cdots\cup[n_{k_i}-n_{k_{i-1}}, n_{k_i}]\cup\cdots$. It is easy to verify that $n_{k_i}-n_{k_{i-1}}>n_{k_i-1}$, so that the only term of the sequence $\{n_k\}$ lying in the interval $[n_{k_i}-n_{k_{i-1}}, n_{k_i}]$ is n_{k_i}. Hence $\hat{g}(n_k)=0$ for all the terms of $\{n_k\}$ except the n_{k_i}. Thus we can solve the interpolation problem for each of the q subsequences and add the solutions to obtain an H^∞ function which solves the interpolation problem for $\{n_k\}$ and $\{v_k\}$. This completes the proof.

We can sharpen the conclusion of the Theorem in two standard ways. First, the function g in H^∞ can be chosen so that $\|g\|_\infty\leqq(qe)^{1/2}\|v\|_2$, where $\|v\|_2=(\sum_{k=0}^\infty |v_k|^2)^{1/2}$ and q is the integer defined above. To see this we begin with the basic case when $\delta\geqq 1$ and $q=1$. There is nothing to prove if $\|v\|_2=0$; so let $\|v\|_2>0$. Let g be the weak star limit point of the polynomials g_k. By (a)

$$\|g\|_\infty \leqq \prod_{k=0}^{\infty} (1+|v_k|^2)^{1/2} < \exp(\|v\|_2^2/2).$$

Given $\{v_k\}$, define a new sequence $\{v_k'\}$ by $v_k'=v_k/\|v\|_2$. Applying our method to the sequence $\{v_k'\}$ we obtain a function g' in H^∞ with $\hat{g}'(n_k)=v_k'$ for all k and $\|g'\|_\infty<e^{1/2}$. Let $g=\|v\|_2 g'$. Then $\hat{g}(n_k)=v_k$ for all k and $\|g\|_\infty<e^{1/2}\|v\|_2$, as required. When $\delta<1$, split $\{n_k\}$ into q subsequences and apply this trick to each subsequence. Adding the resulting H^∞ functions gives a function g in H^∞ with $\hat{g}(n_k)=v_k$ for all k. It follows from the Cauchy-Schwarz inequality that $\|g\|_\infty\leqq(qe)^{1/2}\|v\|_2$.

Second, we can arrange that the interpolating H^∞ function g be continuous. To do this choose a function f in L^1 whose Fourier coefficients tend to 0 so slowly that $\sum_{k=0}^\infty |v_k/\hat{f}(n_k)|^2<\infty$; given that $\|v\|_2<\infty$, this is possible by [**5**, Theorem 4.1, p. 22] or [**12**, vol. I, Theorem 1.5, p. 183]. Apply the Theorem to the sequence $\{v_k/\hat{f}(n_k)\}_{k=0}^\infty$ obtaining a function g' in H^∞ with $\hat{g}(n_k)=v_k/\hat{f}(n_k)$ for all k. Then $g=f*g'$, the convolution of f and g', is a continuous H^∞ function with $\hat{g}(n_k)=v_k$ for all k.

2. Let us say that a set of integers $\{n_k\}_{k=0}^\infty$ has the L^∞ *interpolation property* if, for every square summable sequence $\{v_k\}_{k=0}^\infty$, there is a function g in L^∞ with $\hat{g}(n_k)=v_k$ for all k. Similarly, we shall say that a set of nonnegative integers has the H^∞ *interpolation property* if the set has the L^∞ interpolation property and the interpolating function g can always be chosen to be in H^∞. Finally, we shall call a set of nonnegative integers $\{n_k\}_{k=0}^\infty$ a *Hadamard set* if $n_{k+1}>(1+\delta)n_k$ for some $\delta>0$ for all k. We have just shown that every Hadamard set has the H^∞ interpolation property.

There are other ways to prove this fact, which do not give as much information about the interpolating function g as our method does, but which allow a weakening of the assumption on the sequence $\{n_k\}$. First, one can use a method of Paley [6, Lemma 2, pp. 124–126] to show that if a set of a set of nonnegative integers has the L^∞ interpolation property then it has the H^∞ interpolation property. It is well known [8, p. 225] that a set of integers $\{n_k\}$ has the L^∞ interpolation property if and only if it is a $\Lambda(2)$ set, that is if and only if every L^1 function whose transform is supported by $\{n_k\}$ is actually in L^2. Thus, in order that a set of nonnegative integers have the H^∞ interpolation property it is necessary and sufficient that this set be a $\Lambda(2)$ set. In fact, S. A. Vinogradov has shown that one can simultaneously solve the Rudin-Carleson interpolation problem and the above interpolation problem [11].

Unfortunately no arithmetic characterization of $\Lambda(2)$ sets is known. It is known, however, that any union of finitely many Hadamard sets is a $\Lambda(2)$ set and that some $\Lambda(2)$ sets are not finite unions of Hadamard sets [8, p. 210]. Thus, in §1, the hypothesis on $\{n_k\}$ is stronger than necessary. But the proof given in §1 is more elementary than those discussed above.

The first proof that every Hadamard set has the L^∞ interpolation property was indirect [1, Satz I, p. 212]. Subsequently, Salem and Zygmund used Riesz products to give a simple construction of L^∞ interpolating functions [9]. We want to compare their construction with the one given here. For simplicity suppose that $n_{k+1} \geqq 3n_k$ for all k; let $\sum_{k=0}^\infty |v_k|^2 < \infty$. Consider the product

$$\prod_{k=0}^\infty (1 + v_k z^{n_k} - \bar{v}_k z^{-n_k}) = \prod_{k=0}^\infty [1 + 2i \operatorname{Im}(v_k z^{n_k})].$$

Expanding this product formally yields a trigonometric series which can be shown to be the Fourier series of a function f in L^∞ with $\hat{f}(n_k) = v_k$ for all k and $\|f\|_\infty^2 \leqq \prod_{k=0}^\infty (1+4|v_k|^2)$, [9] or [12, vol. I, p. 211]. The support of $\hat{f}$ is contained in the set F of all integers n having the form $n = \pm n_{k_m} \pm n_{k_{m-1}} \pm \cdots \pm n_{k_1}$, where $k_1 < k_2 < \cdots < k_m$. The function f is in H^∞ only if $\|v\|_2 = 0$.

On the other hand, it was shown in [3, Remark 8] that, under the same hypotheses on $\{n_k\}$ and $\{v_k\}$, there is a function g in H^∞ with $\hat{g}(n_k) = v_k$ for all k and with the support of $\hat{g}$ contained in the set G of all integers n having the form $n = n_{k_m} - n_{k_{m-1}} + n_{k_{m-2}} - \cdots - n_{k_2} + n_{k_1}$, where the signs alternate, m is odd, and $k_1 < k_2 < \cdots < k_m$. The proof in [3] does not indicate how to construct such a function g. Since the set G is contained in the set F associated with the Riesz product f it is natural to consider the series $\sum_{n \in G} \hat{f}(n) z^n$ and attempt to show that this is the Fourier series of a bounded function. In fact it is the Fourier series of the function g constructed in §1.

It is not clear whether this relation between g and f can be used to prove that g is bounded. Instead we have modified a method due independently to Shapiro [**10**, p. 35] and Rudin [**7**]. The idea of setting up two sequences of trigonometric polynomials with prescribed coefficients is due to them. What is new here is the use of the identity (1), rather than the parallelogram law, to estimate the L^∞ norms of the polynomials; this change allows us to generate polynomials whose nonzero coefficients do not all have the same absolute value.

The method used here seems to be dual to the Hilbert space method of [**3**, Lemma 2], but has the advantage of producing the interpolating function g explicitly. Otherwise the two methods are equivalent in the sense that anything which can be proved by one of the methods can also be proved by the other method. For instance, if $n_{k+1}>2n_k$ for all k, and $v_k\neq 0$ for all k, then the function g constructed in §1 has the property that the support of $\hat{g}$ is exactly the set G defined above, and in any case $\hat{g}$ is carried by G; it was shown in [**3**, Remark 8] that there must be an interpolating function whose transform is carried by G. Because of this equivalence many of the results in [**3**] can now be proved in two ways. In particular we now have two proofs of Theorem 1 of [**3**] but we still do not know whether the assumption that l is an exponential gap system implies that $\sum_{k=0}^\infty |\hat{\mu}(m_k)|^2<\infty$ for all μ in M_l; see [**3**, §2] for definitions of these terms.

For further applications of the method of this paper to problems in several variables, see [**2**].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER 8, BRITISH COLUMBIA, CANADA