THE DISCRETENESS OF THE SPECTRUM OF SELF-ADJOINT, EVEN ORDER, ONE-TERM, DIFFERENTIAL OPERATORS

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Abstract. An open question which was asked by I. M. Glazman is answered. It is well known that the condition
\[ \lim_{x \to \infty} x^{2n-1} \int_{x}^{\infty} r^{-1} = 0 \]
is sufficient for the discreteness and boundedness from below of the spectrum of selfadjoint extensions of \((-1)^n(y^{(n)}(x))^n\). This paper shows that the condition is also necessary.

Let \( \tilde{L} \) denote any selfadjoint extension of \((-1)^n(y^{(n)}(x))^n \quad (x \geq 0, r(x) > 0)\).

Theorem 1. A necessary and sufficient condition that the spectrum of \( \tilde{L} \) be discrete and bounded below is that
\[ \lim_{x \to \infty} x^{2n-1} \int_{x}^{\infty} r^{-1} = 0. \]

The proof of the sufficiency part of the above theorem is due to V. A. Tkachenko and is exhibited in a book by I. M. Glazman [2, pp. 120, 121]. Glazman states that if \( r(x) \) is monotonic, then relation (1) is necessary for the discreteness of the spectrum, but in the general case the problem of the necessity of condition (1) for \( n > 1 \) remains open. The development below completes the proof of Theorem 1.

Define
\[ L_{2n}y = \sum_{k=0}^{n} (-1)^{n-k}(p_k y^{(n-k)}(x))^{(n-k)} \]
where we assume that \( p_k \) is \( n-k \) times continuously differentiable. The domain of \( L_{2n} \) is the set of all \( y \) such that \( y \) and \( L_{2n}y \) are elements of \( L^2[a, b] \) for \( a, b \) satisfying \( 0 < a < b < \infty \) and \( y^{(k)} \) is absolutely continuous on compact subintervals of \( (0, \infty) \) for \( k = 0, 1, \ldots, 2n-1 \). Let \( \mathcal{D} \) denote...
the set of all $y$ in the domain of $L_{2n}$ such that $y$ has compact support interior to $[a, \infty)$ for some $a>0$. Let $T$ denote the closure of the restriction of $L_{2n}$ to $\mathcal{D}$.

The differentiable operator $L_{2n}$ is said to be oscillatory on $[a, \infty)$ if for any given $b>a$ there are numbers $c$ and $d$ and a function $y \neq 0$ such that $L_{2n}(y)=0$, $b \leq c < d$, and

$$y^{(k)}(c) = 0 = y^{(k)}(d) \quad \text{for} \quad k = 0, 1, \ldots, n-1.$$ 

Otherwise, $L_{2n}$ is said to be nonoscillatory on $[a, \infty)$.

**Theorem 2 (Glazman [2]).** The following statements are equivalent.

(i) The spectrum of every selfadjoint extension of $T$ is bounded below and discrete.

(ii) For every real number $\lambda$, $L_{2n} - \lambda$ is nonoscillatory on $[0, \infty)$.

**Theorem 3 (Ahlbrandt [1]).** Suppose that $r$ and $p$ are positive, real-valued functions which are Lebesgue integrable on arbitrary compact subintervals of $[0, \infty)$. Then $(-1)^n(r^{-1}y^{(n)})^{(n)} + py$ is nonoscillatory on $[a, \infty)$ for some $a>0$ if and only if $(-1)^n(p^{-1}y^{(n)})^{(n)} - ry$ is nonoscillatory on $[a, \infty)$.

The following theorem is an extension by the author [3] of a theorem due to Glazman [2].

**Theorem 4.** If $p(x) \leq 0$, $0 < r(x) \leq M x^\alpha$ for some $\alpha < 2n-1$, and

$$\limsup_{x \to \infty} x^{2n-1-x} \int_x^\infty |p| > M \cdot A_n^2$$

where

$$A_n^{-1} = \frac{(2n-1)^{1/2}}{(n-1)!} \sum_{k=1}^n (-1)^{k-1} \binom{n-1}{k-1} (2n-k)$$

then $(-1)^n(r y^{(n)})^{(n)} + py$ is oscillatory on $[a, \infty)$ for $a>0$.

**Proof of Theorem 1.** (Necessity) Suppose that

$$\lim_{x \to \infty} x^{2n-1} \int_x^\infty r^{-1} \neq 0;$$

then

$$\limsup_{x \to \infty} x^{2n-1} \int_x^\infty r^{-1} = \beta > 0.$$ 

There is a constant $c>0$ such that $\lambda \geq c$ implies that $\beta > A_n^2/\lambda$. Hence, by Theorem $4$, $(-1)^n(\lambda^{-1} y^{(n)})^{(n)} - r^{-1} y$ is oscillatory on $[a, \infty)$ for $a>0$ and

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\lambda \geq c$ which implies that \((-1)^n (ry^{(n)})^{(n)} - \lambda y\) is oscillatory on \([a, \infty)\) by Theorem 3. Consequently, by Theorem 2, the spectrum of \(L\) is not necessarily discrete and bounded below.

References


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