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THE DISCRETENESS OF THE SPECTRUM OF SELF-ADJOINT, EVEN ORDER, ONE-TERM, DIFFERENTIAL OPERATORS

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ABSTRACT. An open question which was asked by I. M. Glazman is answered. It is well known that the condition

$$\lim_{x\to\infty} x^{2n-1}\int_x^\infty r^{-1} = 0$$

is sufficient for the discreteness and boundedness from below of the spectrum of selfadjoint extensions of $(-1)^n(ry^{(n)})^{(n)}$. This paper shows that the condition is also necessary.

Let $\tilde{L}$ denote any selfadjoint extension of

$$(-1)^n(ry^{(n)})^{(n)} \qquad (x \geqq 0,\ r(x) > 0).$$

THEOREM 1. *A necessary and sufficient condition that the spectrum of $\tilde{L}$ be discrete and bounded below is that*

$$\lim_{x\to\infty} x^{2n-1}\int_x^\infty r^{-1} = 0. \tag{1}$$

The proof of the sufficiency part of the above theorem is due to V. A. Tkachenko and is exhibited in a book by I. M. Glazman [**2**, pp. 120, 121]. Glazman states that if $r(x)$ is monotonic, then relation (1) is necessary for the discreteness of the spectrum, but in the general case the problem of the necessity of condition (1) for $n>1$ remains open. The development below completes the proof of Theorem 1.

Define

$$L_{2n}y = \sum_{k=0}^{n} (-1)^{n-k}(p_k y^{(n-k)})^{(n-k)}$$

where we assume that p_k is $n-k$ times continuously differentiable. The domain of L_{2n} is the set of all y such that y and $L_{2n}y$ are elements of $L^2[a, b]$ for a, b satisfying $0<a<b<\infty$ and $y^{(k)}$ is absolutely continuous on compact subintervals of $(0, \infty)$ for $k=0, 1, \cdots, 2n-1$. Let $\mathscr{D}$ denote

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the set of all y in the domain of L_{2n} such that y has compact support interior to $[a, \infty)$ for some $a>0$. Let T denote the closure of the restriction of L_{2n} to $\mathcal{D}$.

The differentiable operator L_{2n} is said to be *oscillatory on* $[a, \infty)$ if for any given $b>a$ there are numbers c and d and a function $y\not\equiv 0$ such that $L_{2n}(y)=0$, $b\leqq c<d$, and

$$y^{(k)}(c) = 0 = y^{(k)}(d) \quad \text{for } k = 0, 1, \cdots, n-1.$$

Otherwise, L_{2n} is said to be *nonoscillatory on* $[a, \infty)$.

THEOREM 2 (GLAZMAN [2]). *The following statements are equivalent.*

(i) *The spectrum of every selfadjoint extension of T is bounded below and discrete.*

(ii) *For every real number* λ, $L_{2n}-\lambda$ *is nonoscillatory on* $[0, \infty)$.

THEOREM 3 (AHLBRANDT [1]). *Suppose that r and p are positive, real-valued functions which are Lebesgue integrable on arbitrary compact subintervals of* $[0, \infty)$. *Then* $(-1)^n(r^{-1}y^{(n)})^{(n)}-py$ *is nonoscillatory on* $[a, \infty)$ *for some* $a>0$ *if and only if* $(-1)^n(p^{-1}y^{(n)})^{(n)}-ry$ *is nonoscillatory on* $[a, \infty)$.

The following theorem is an extension by the author [3] of a theorem due to Glazman [2].

THEOREM 4. *If* $p(x)\leqq 0$, $0<r(x)\leqq Mx^\alpha$ *for some* $\alpha<2n-1$, *and*

$$\limsup_{x\to\infty} x^{2n-1-\alpha}\int_x^\infty |p| > M\cdot A_n^2$$

where

$$A_n^{-1} = \frac{(2n-1)^{1/2}}{(n-1)!}\sum_{k=1}^{n}(-1)^{k-1}\binom{n-1}{k-1}\Big/(2n-k)$$

then $(-1)^n(ry^{(n)})^{(n)}+py$ *is oscillatory on* $[a, \infty)$ *for* $a>0$.

PROOF OF THEOREM 1. (*Necessity*) Suppose that

$$\lim_{x\to\infty} x^{2n-1}\int_x^\infty r^{-1} \neq 0;$$

then

$$\limsup_{x\to\infty} x^{2n-1}\int_x^\infty r^{-1} = \beta > 0.$$

There is a constant $c>0$ such that $\lambda\geqq c$ implies that $\beta>A_n^2/\lambda$. Hence, by Theorem 4,[1] $(-1)^n(\lambda^{-1}y^{(n)})^{(n)}-r^{-1}y$ is oscillatory on $[a, \infty)$ for $a>0$ and

[1] Theorem 11/31 (9)/ , p. 100 of Glazman [2] could also be used if one lets $q=-\lambda r^{-1}$.

$\lambda \geqq c$ which implies that $(-1)^n(ry^{(n)})^{(n)} - \lambda y$ is oscillatory on $[a, \infty)$ by Theorem 3. Consequently, by Theorem 2, the spectrum of $\tilde{L}$ is not necessarily discrete and bounded below.

References

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