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ASYMPTOTICS OF STIRLING NUMBERS OF THE SECOND KIND

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ABSTRACT. A complete asymptotic development of the Stirling numbers $S(N, K)$ of the second kind is obtained by the saddle point method previously employed by Moser and Wyman [Trans, Roy. Soc. Canad., **49** (1955), 49–54] and de Bruijn [*Asymptotic methods in analysis*, North-Holland, Amsterdam, 1958, pp. 102–109] for the asymptotic representation of the related Bell numbers.

1. **Introduction.** Hsu [1] has given the asymptotic expansion

$$(1)\quad S(N,K) \sim (\tfrac{1}{2}K^2)^{N-K}\left[1 + \sum_{s=1}^{t} K^{-s} f_s(N-K) + O(K^{-t-1})\right] \Big/ (N-K)!$$

for Stirling numbers $S(N, K)$ of the second kind, where $f_s(N-K)$ are polynomials of argument $N-K$ and $f_s(0)=0$. The expansion (1) is useful only for $N-K$ small, as is indicated in §3. We obtain a complete asymptotic expansion of $S(N, K)$ in powers of $(N+1)^{-1}$, using the saddle point method previously employed by Moser and Wyman [2] and de Bruijn [3] for the asymptotic representation of the related Bell numbers. Convergence is demonstrated for $K<(N+1)^{2/3}/[\pi+(N+1)^{-1/3}]$. The expansion when divergent is still useful when used as an asymptotic series.

2. **Asymptotics of** $S(N, K)$. A generating function for $S(N, K)$ is

$$(2)\qquad \left(\frac{e^z-1}{z}\right)^K = \sum_{N=K}^{\infty} \frac{K!}{N!} S(N, K) z^{N-K}.$$

Hence the Cauchy integral formula gives

$$(3)\qquad S(N, K) = \frac{N!}{2\pi i K!}\int_C (e^z-1)^K z^{-N-1}\, dz$$

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where the contour C encloses the origin. Equating the derivative of the integrand to zero gives the equation

$$(t - z)e^{z-t} = te^{-t}, \tag{4}$$

where $t=(N+1)/K$, for the location of the saddle point of the modulus of the integrand. The principal saddle point $z=u$ is on the positive real axis with $t-1<u<t$. The quadratic approximation to xe^{-x} at $x=1$ shows that $u\approx 2/N$ for $K=N$ and large N. There are other subsidiary saddle points at complex roots of (4), which we neglect in comparison with the higher saddle point at $z=u$. Since there are no other roots of (4) for $|t-z|\leqq t-u$, we may apply the Lagrange inversion formula to obtain

$$u = t - \sum_{m=1}^{\infty} m^{m-1}(te^{-t})^m/m! \tag{5}$$

convergent for $t>1$. Another form of (4) is the identity

$$K = (N + 1)(1 - e^{-u})/u \tag{6}$$

needed later. Since the axis of the saddle point is perpendicular to the real axis, the part of the contour C descending from $z=u$ is taken as the line $z=u+iy$, $|y|<\infty$, parallel to the imaginary axis. The modulus of the integrand in (3) is maximal at $z=u$ on this path, since both $(e^z-1)^K$ and z^{-N-1} have this property. The closed contour C is completed by a half circle of infinite radius enclosing the origin. The contribution to the integral (3) on this semicircular path is zero since $N>0$. The integral in (3) then becomes

$$i(e^u - 1)^K u^{-N-1}\int_{-\infty}^{\infty} \exp \psi(u + iy)\, dy \tag{7}$$

where

$$\psi(z) = K \ln[(e^z - 1)/(e^u - 1)] - (N + 1)\ln(z/u). \tag{8}$$

The contribution of the various parts of the $z=u+iy$ path to the integral must now be studied. As $|\exp \psi(z)|=\exp \operatorname{Re} \psi(z)$ we have to study

$$\begin{aligned}\operatorname{Re} \psi(u + iy) = K \ln[(e^{2u} - 2e^u \cos y+1)^{1/2}/(e^u - 1)]\\ -(N + 1)\ln(1 + y^2u^{-2})^{1/2}.\end{aligned}$$

We shall show that we can restrict ourselves essentially to the interval $|y|<\pi$. Since $1+y^2u^{-2}\geqq 1+\pi(2y-\pi)u^{-2}$ for $y\geqq\pi$ we have

$$(e^u - 1)^K u^{-N-1}\left|\int_{\pi}^{\infty} \exp \psi(u + iy)\, dy\right| < \frac{u^{1-N}(e^u + 1)^K}{\pi(N - 1)(1 + \pi^2u^{-2})^{N/2-1/2}}$$

which is of $O(N^{-N}e^N)$ for K small and of $O(2^N/\pi^N N)$ for K large. Since $\operatorname{Re}\psi(u+iy)$ is even, the part of the integral (7) for $|y|>\pi$ tends toward zero as $N\to\infty$. We now direct our attention to the interval $|y|<\pi$ where the saddle point at $y=0$ gives the main contribution. The Taylor expansion of $\psi(u+iy)$, convergent for $|y|<u$, is

$$\begin{aligned}\psi = &-\frac{N+1}{2u}\left(\frac{1}{u}-\frac{1}{e^u-1}\right)y^2 \\ &+(N+1)\sum_{j=1}^{\infty}\frac{(iy)^{j+2}}{(j+2)!}\left(\frac{d}{dz}\right)^{j+1}\left[\frac{1-e^{-u}}{u(e^z-1)}-\frac{1}{z}\right]_{z=u}\end{aligned} \tag{9}$$

where the identity (6) has been used. We now make the substitutions

$$w = [(N+1)/2]^{1/2}[1-u/(e^u-1)]^{1/2}y/u \tag{10}$$

and

$$a_j = \frac{(iwu)^{j+2}(d/dz)^{j+1}[(1-e^{-u})/u(l^z-1)-1/z]_{z=u}}{(j+2)!\,[\frac{1}{2}-\frac{1}{2}u/(e^u-1)]^{j/2+1}} \tag{11}$$

to obtain

$$S(N,K) = B\int_{-\infty}^{\infty}\exp\{-w^2+f[(N+1)^{-1/2}]\}\,dw \tag{12}$$

where

$$B = N!(e^u-1)^K/\pi(2(N+1))^{1/2}K!\,u^N(1+u/(1-\exp u))^{1/2} \tag{13}$$

and f is the analytic continuation of

$$f[(N+1)^{-1/2}] = \sum_{j=1}^{\infty} a_j(N+1)^{-j/2}. \tag{14}$$

To find an upper bound to $|a_j|$ we note that $(e^z-1)^{-1}=\sum_{k=1}^{\infty}e^{-kz}$ for $\operatorname{Re} z>0$. Then

$$(d/du)^n(e^u-1)^{-1} = (-1)^n\sum_{x=0}^{\infty}g(x)$$

where $g(x)=x^ne^{-ux}$. On using the Euler-Maclaurin sum formula **[5]** we find

$$\sum_{x=0}^{\infty}g(x) = \frac{n!}{u^{n+1}}+R_n.$$

The remainder $R_n=\int_0^\infty (x-[x]-\frac{1}{2})g'(x)\,dx$ may be evaluated by a Laplace transform **[6]** to be

$$R_n = (-1)^{n-1}[nF^{(n-1)}(u)+uF^{(n)}(u)]$$

where $F(u)=u^{-2}-\frac{1}{2}u^{-1}\coth\frac{1}{2}u$. We conclude that $|R_n|\ll n!/u^{n+1}$ for small u and tends to zero for large u. Since $(1-e^{-u})/u$ is less than unity we have

$$|a_j| < \sigma^{j+2}/j \tag{15}$$

where

$$\sigma = w2^{1/2}/(1-u/(e^u-1))^{1/2} = y(N+1)^{1/2}/u. \tag{16}$$

Remembering that we need not integrate (7) beyond $|y|=\pi$ for large N, we see by (15) and (16) that the series (14) is convergent for $\pi/u<1$. We now expand $\exp f[(N+1)^{-1/2}]$ in a Taylor series of the form

$$\exp f[(N+1)^{-1/2}] = \sum_{j=0}^{\infty} b_j(N+1)^{-j/2} \tag{17}$$

where $b_0=1$ and b_j are polynomials in w of the degree and parity of $3j$. By a lemma of Moser and Wyman [4]

$$|b_j| \leqq \sigma^{j+2}(1+\sigma^2)^{j-1}. \tag{18}$$

Using (17) we may write (12) in the form

$$S(N,K) = B\left[\sum_{j=0}^{s-1}(N+1)^{-j}\int_{-\infty}^{\infty} e^{-w^2}b_{2j}\,dw + R_s\right].$$

The absolute value of the remainder R_s is found from (18) to be

$$|R_s| \leqq (N+1)^{-s}\int_{-\infty}^{\infty} e^{-w^2}P_s(|w|)\,dw/M \tag{19}$$

where $P_s(|w|)$ is a polynomial in $|w|$ and

$$M = 1-\sigma^2(1+\sigma^2)^2/(N+1).$$

On limiting the integration in (7) to $|y|<\pi$ we see that the remainder R_s exists if

$$(\pi/u)[1+(N+1)(\pi/u)^2] < 1.$$

Since $u+1>(N+1)/K$ convergence occurs for

$$K < (N+1)^{2/3}/[\pi+(N+1)^{-1/3}]$$

approximately. For these values of K we conclude that

$$S(N,K) \sim B\left\{\sum_{j=0}^{s-1}(N+1)^{-j}\int_{-\infty}^{\infty} e^{-w^2}b_{2j}\,dw + O[(N+1)^{-s}]\right\}. \tag{20}$$

The first two terms of (20) have been calculated to be

$$S(N,K) \sim \frac{N!(e^u-1)^K}{(2\pi(N+1))^{1/2}K!\,u^N(1-G)^{1/2}} \tag{21}$$
$$\cdot\left[1-\frac{2+18G-20G^2(e^u+1)}{24(N+1)(1-G)^3} + \frac{3G^3(e^{2u}+4e^u+1)+2G^4(e^{2u}-e^u+1)}{24(N+1)(1-G)^3}\right]$$

where by [5]

$$G = u/(e^u-1) = 1-\tfrac{1}{2}u+\sum_{k=1}^{\infty} B_{2k}u^{2k}/(2k)!.$$

The bracketed expression in (21), argumented by an additional inverse power of $N+1$, is approximated by

$$1-\frac{1}{6u(N+1)}+\frac{1}{72u^2(N+1)^2}$$

for small u and by

$$1-\frac{1}{12(N+1)}+\frac{1}{288(N+1)^2}$$

for large u. These are the leading terms of an alternating asymptotic series.

3. **Numerical example.** The 6-significant-figure Table 1 compares the exact values of $S(100, K)$ with the values computed from (20) and (1) for several K. The excellent results obtained from (20) for values of K outside the interval of convergence show that the expansion gives useful results when used as an asymptotic series.

TABLE 1

K	$S(100, K)$ Exact	$S(100, K)$ 1 term of (20)	$S(100, K)$ 2 terms of (20)	$S(100, K)$ 4 terms of (1)
2	6.33825 10^{29}	6.34348 10^{29}	6.33825 10^{29}	1.81186 10^{-115}
25	2.58320 10^{114}	2.58496 10^{114}	2.58321 10^{114}	2.94696 10^{83}
50	4.30983 10^{101}	4.30900 10^{101}	4.30977 10^{101}	1.51529 10^{94}
75	1.82584 10^{63}	1.82671 10^{63}	1.82579 10^{63}	5.32626 10^{62}
99	4.95000 10^{3}	5.14199 10^{3}	4.94451 10^{3}	4.95000 10^{3}

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