ASYMPTOTICS OF STIRLING NUMBERS OF THE SECOND KIND
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1. Introduction. Hsu [1] has given the asymptotic expansion

\[
S(N, K) \sim \left(\frac{1}{\sqrt{2\pi}}\right)^N N^{-K} \left(1 + \sum_{s=1}^{\lfloor N/2 \rfloor} K^{-s} f_s(N - K) + O(K^{-s-1})\right)(N - K)!
\]

for Stirling numbers \( S(N, K) \) of the second kind, where \( f_s(N - K) \) are polynomials of argument \( N - K \) and \( f_s(0) = 0 \). The expansion (1) is useful only for \( N - K \) small, as is indicated in §3. We obtain a complete asymptotic expansion of \( S(N, K) \) in powers of \( (N+1)^{-1} \), using the saddle point method previously employed by Moser and Wyman [2] and de Bruijn [3] for the asymptotic representation of the related Bell numbers. Convergence is demonstrated for \( K < (N+1)^{2/3}/[\pi + (N+1)^{-1/3}] \). The expansion when divergent is still useful when used as an asymptotic series.

2. Asymptotics of \( S(N, K) \). A generating function for \( S(N, K) \) is

\[
\left(\frac{e^z - 1}{z}\right)^K = \sum_{N=K}^{\infty} \frac{K!}{N!} S(N, K) z^{N-K}.
\]

Hence the Cauchy integral formula gives

\[
S(N, K) = \frac{N!}{2\pi i} \int_C (e^z - 1)^K z^{-N-1} \, dz
\]
where the contour $C$ encloses the origin. Equating the derivative of the integrand to zero gives the equation

$$(4) \quad (t - z)e^{z-t} = te^{-t},$$

where $t=(N+1)/K$, for the location of the saddle point of the modulus of the integrand. The principal saddle point $z=u$ is on the positive real axis with $t-1<u<t$. The quadratic approximation to $xe^{-x}$ at $x=1$ shows that $u=2/N$ for $K=N$ and large $N$. There are other subsidiary saddle points at complex roots of (4), which we neglect in comparison with the higher saddle point at $z=u$. Since there are no other roots of (4) for $|t-z|\leq t-u$, we may apply the Lagrange inversion formula to obtain

$$(5) \quad u = t - \sum_{m=1}^{\infty} m^{m-1} (te^{-t})^m/m!$$

convergent for $t>1$. Another form of (4) is the identity

$$(6) \quad K = (N+1)(1 - e^{-u})/u$$

needed later. Since the axis of the saddle point is perpendicular to the real axis, the part of the contour $C$ descending from $z=u$ is taken as the line $z=u+iy$, $|y|<\infty$, parallel to the imaginary axis. The modulus of the integrand in (3) is maximal at $z=u$ on this path, since both $(e^z-1)^K$ and $z-a-i$ have this property. The closed contour $C$ is completed by a half circle of infinite radius enclosing the origin. The contribution to the integral (3) on this semicircular path is zero since $N>0$. The integral in (3) then becomes

$$(7) \quad i(e^u - 1)^K u^{-N-1} \int_{-\infty}^{\infty} \exp \psi(u + iy) \, dy$$

where

$$(8) \quad \psi(z) = K \ln[(e^{z+1})/(e^u - 1)] - (N + 1)\ln(z/u).$$

The contribution of the various parts of the $z=u+iy$ path to the integral must now be studied. As $|\exp \psi(z)|=\exp \Re \psi(z)$ we have to study

$$\Re \psi(u + iy) = K \ln[(e^{2u} - 2e^u \cos y+1)^{1/2}/(e^u - 1)] - (N + 1)\ln(1 + y^2u^2)^{1/2}.$$ 

We shall show that we can restrict ourselves essentially to the interval $|y|<\pi$. Since $1+y^2u^{-2}\geq1+\pi(2y-\pi)u^{-2}$ for $y\geq\pi$ we have

$$(e^u - 1)^K u^{-N-1} \int_{\pi}^{\infty} \exp \psi(u + iy) \, dy < \frac{u^{1-N}(e^u + 1)^K}{\pi(N - 1)(1 + \pi^2u^{-2})^{N/2-1/2}}$$
which is of $O(N^{-N}e^N)$ for $K$ small and of $O(2^N/\pi^N N)$ for $K$ large. Since $\Re \psi(u+iy)$ is even, the part of the integral (7) for $|y|>\pi$ tends toward zero as $N \to \infty$. We now direct our attention to the interval $|y|<\pi$ where the saddle point at $y=0$ gives the main contribution. The Taylor expansion of $\psi(u+iy)$, convergent for $|y|<u$, is

\[ \psi = -\frac{N+1}{2u} \left( \frac{1}{u} - \frac{1}{e^u - 1} \right) y^2 + (N+1) \sum_{j=1}^{\infty} \frac{(1 + iy)^j}{(j+2)!} \left( \frac{d}{dz} \right)^{j+1} \left[ \frac{1 - e^{-u}}{u(e^u - 1)} - \frac{1}{z} \right]_{z=u} \]

where the identity (6) has been used. We now make the substitutions

\[ w = [(N+1)/2]^{1/2} \left[ 1 - u/(e^u - 1) \right]^{1/2} y/|u| \]

and

\[ a_j = \frac{(i w)^{j+2} (d/dz)^{j+1} \left[ (1 - e^{-u})/u(e^u - 1) - 1/|z| \right]_{z=u}}{(j+2)! \left[ \frac{1}{2} - \frac{1}{2} u/(e^u - 1) \right]^{j+1}} \]

to obtain

\[ S(N, K) = B \int_{-\infty}^{\infty} \exp\left\{ -w^2 + f[(N+1)^{-1/2}] \right\} \, dw \]

where

\[ B = N! (e^u - 1)^{K/2}(2(N+1))^{1/2} K! u^{N} (1 + u/(-\exp u))^{1/2} \]

and $f$ is the analytic continuation of

\[ f[(N+1)^{-1/2}] = \sum_{j=1}^{\infty} a_j (N+1)^{-j/2}. \]

To find an upper bound to $|a_j|$ we note that $(e^x - 1)^{-1} = \sum_{k=1}^{\infty} e^{-kx}$ for $\Re x > 0$. Then

\[ (d/du)^{n}(e^u - 1)^{-1} = (-1)^n \sum_{z=0}^{\infty} g(x) \]

where $g(x) = x^ne^{-ux}$. On using the Euler-Maclaurin sum formula [5] we find

\[ \sum_{x=0}^{\infty} g(x) = \frac{n!}{u^{n+1}} + R_n. \]

The remainder $R_n = \int_{0}^{\infty} (x-[x]-\frac{1}{2})g'(x) \, dx$ may be evaluated by a Laplace transform [6] to be

\[ R_n = (-1)^{n-1} [nF^{(n-1)}(u) + uF^{(n)}(u)] \]
where \( F(u) = u^{-2} - \frac{1}{2} u^{-1} \coth \frac{1}{2} u \). We conclude that \(|R_n| \ll n! / u^{n+1}\) for small \( u \) and tends to zero for large \( u \). Since \((1 - e^{-u}) / u \) is less than unity we have

\[
|a_j| < \sigma^{j+2}/j
\]

where

\[
\sigma = w^{2^{1/2}}(1 - u/(e^u - 1))^{1/2} = y(N + 1)^{1/2}/u.
\]

Remembering that we need not integrate (7) beyond \(|y| = \pi\) for large \( N \), we see by (15) and (16) that the series (14) is convergent for \( \pi/|u| < 1 \). We now expand \( \exp f[(N+1)^{-1/2}] \) in a Taylor series of the form

\[
\exp f[(N + 1)^{-1/2}] = \sum_{j=0}^{\infty} b_j(N + 1)^{-j/2}
\]

where \( b_0 = 1 \) and \( b_j \) are polynomials in \( w \) of the degree and parity of \( 3j \).

By a lemma of Moser and Wyman [4]

\[
|b_j| \leq \sigma^{j+2}(1 + \sigma^2)^{j-1}.
\]

Using (17) we may write (12) in the form

\[
S(N, K) = B \left( \sum_{j=0}^{\infty} (N + 1)^{-j} \int_{-\infty}^{\infty} e^{-w^2} b_{2j} dw + R_s \right).
\]

The absolute value of the remainder \( R_s \) is found from (18) to be

\[
|R_s| \leq (N + 1)^{-j} \int_{-\infty}^{\infty} e^{-w^2} P_s(|w|) dw / M
\]

where \( P_s(|w|) \) is a polynomial in \(|w|\) and

\[
M = 1 - \sigma^2(1 + \sigma^2)^2/(N + 1).
\]

On limiting the integration in (7) to \(|y| < \pi\) we see that the remainder \( R_s \) exists if

\[
(\pi/|u|)[1 + (N + 1)(\pi/|u|^2)] < 1.
\]

Since \( u + 1 > (N + 1)/K \) convergence occurs for

\[
K < (N + 1)^{2/3} / [\pi + (N + 1)^{-1/3}]
\]

approximately. For these values of \( K \) we conclude that

\[
S(N, K) \sim B \left( \sum_{j=0}^{\infty} (N + 1)^{-j} \int_{-\infty}^{\infty} e^{-w^2} b_{2j} dw + O((N + 1)^{-j}) \right).
\]
The first two terms of (20) have been calculated to be

\[ S(N, K) \sim \frac{N!(e^u - 1)^K}{(2\pi(N + 1)^{3/2}K!u^N(1 - G)^{1/2}} \]

\[ \times \left[ 1 - \frac{2 + 18G - 20G^2(e^u + 1)}{24(N + 1)(1 - G)^3} \right. \]

\[ + \left. \frac{3G^3(e^{2u} + 4e^u + 1) + 2G^4(e^{2u} - e^u + 1)}{24(N + 1)(1 - G)^3} \right] \]

where by [5]

\[ G = u/(e^u - 1) = 1 - \frac{u}{2} + \sum_{k=1}^{\infty} \frac{B_{2k} u^{2k}/(2k)!}. \]

The bracketed expression in (21), argumented by an additional inverse power of \( N + 1 \), is approximated by

\[ 1 - \frac{1}{6u(N + 1)} + \frac{1}{72u^3(N + 1)^2} \]

for small \( u \) and by

\[ 1 - \frac{1}{12(N + 1)} + \frac{1}{288(N + 1)^2} \]

for large \( u \). These are the leading terms of an alternating asymptotic series.

3. Numerical example. The 6-significant-figure Table 1 compares the exact values of \( S(100, K) \) with the values computed from (20) and (1) for several \( K \). The excellent results obtained from (20) for values of \( K \) outside the interval of convergence show that the expansion gives useful results when used as an asymptotic series.

<table>
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<tr>
<th>( K )</th>
<th>( S(100, K) ) Exact</th>
<th>( S(100, K) ) 1 term of (20)</th>
<th>( S(100, K) ) 2 terms of (20)</th>
<th>( S(100, K) ) 4 terms of (1)</th>
</tr>
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<tr>
<td>2</td>
<td>6.33825 ( 10^{10} )</td>
<td>6.34348 ( 10^{10} )</td>
<td>6.33825 ( 10^{10} )</td>
<td>1.81186 ( 10^{-11} )</td>
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<tr>
<td>25</td>
<td>2.58320 ( 10^{11} )</td>
<td>2.58496 ( 10^{11} )</td>
<td>2.58321 ( 10^{11} )</td>
<td>2.94696 ( 10^{9} )</td>
</tr>
<tr>
<td>50</td>
<td>4.30983 ( 10^{10} )</td>
<td>4.30900 ( 10^{10} )</td>
<td>4.30977 ( 10^{10} )</td>
<td>1.51529 ( 10^{4} )</td>
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<tr>
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<td>1.82584 ( 10^{9} )</td>
<td>1.82671 ( 10^{9} )</td>
<td>1.82579 ( 10^{9} )</td>
<td>5.32626 ( 10^{4} )</td>
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<td>4.95000 ( 10^{8} )</td>
<td>5.14199 ( 10^{8} )</td>
<td>4.94451 ( 10^{8} )</td>
<td>4.95000 ( 10^{8} )</td>
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REFERENCES


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