MULTIPLIERS FOR $L_1$-ALGEBRAS WITH APPROXIMATE IDENTITIES

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Abstract. Let $S$ be a commutative semigroup with multiplier semigroup $\Omega(S)$. Assume that $l_1(S)$ is semisimple and possesses a bounded approximate identity. If $l_1(\Omega(S))$ denotes the annihilator of $l_1(S)$ in $l_1(\Omega(S))$, then the multiplier algebra of $l_1(S)$ is topologically isomorphic to $l_1(\Omega(S))/l_1(S)^\#$, and this quotient algebra of $l_1(\Omega(S))$ is itself an $L_1$-algebra.

1. Introduction. Let $S$ be a commutative semigroup with multiplier semigroup $\Omega(S)$. If $l_1(S)$ is semisimple and possesses an approximate identity of norm one, then it is proved in [4] that the multiplier algebra $\mathcal{M}(l_1(S))$ is isometrically isomorphic to $l_1(\Omega(S))$. The purpose of this paper is to describe $\mathcal{M}(l_1(S))$ when $l_1(S)$ possesses an approximate identity bounded by some number $R > 1$. The main result is contained in Theorem 4.2 where it is proved that in general $\mathcal{M}(l_1(S))$ is topologically a quotient algebra of $l_1(\Omega(S))$. Interestingly enough, it turns out that this quotient algebra is also an $L_1$-algebra.

The paper is organized as follows: §2 is devoted to notation and background material; in §3 it is shown that it is always possible to imbed a certain subsemigroup of $\Omega(S)$ in the structure semigroup of $l_1(S)$; and finally, §4 contains the main result.

2. Preliminaries. Throughout this paper $S$ is a commutative semigroup, and $(l_1(S), \ast)$ is semisimple. Unless otherwise stated, $l_1(S)$ possesses a bounded approximate identity of norm $R$. That is, there exists a net $\{E_d\} \subseteq l_1(S)$ such that $\|x \ast E_d - x\| \to 0$ for all $x \in l_1(S)$ and $\|E_d\| \leq R$ for all $d$ and some positive number $R$. If $\Gamma$ denotes the structure semigroup associated with $(l_1(S), \ast)$ [6], then there is an isomorphism $i_\ast$ of $S$ onto a dense subsemigroup of $\Gamma$ [3]. Moreover, the fact that $l_1(S)$ has a bounded approximate identity implies that $S$ contains a set of relative units and $\Gamma$ contains a finite set of relative units [4]. Let $U = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be a minimal set of relative units for $\Gamma$, with $\gamma_i^2 = \gamma_i$ for all $i$ [2]; thus, for each
γ ∈ Γ there exists j such that γγj = γ. Further define \( \hat{Γ} \) to be the set of all nonzero continuous semicharacters on Γ.

A bounded linear operator \( T \) from \( l_1(S) \) into \( l_1(S) \) is called a multiplier of \( l_1(S) \) if \( T(α * β) = α * T(β) \) for all \( α, β ∈ l_1(S) \). The set of multipliers of \( l_1(S) \) is a commutative Banach algebra of operators under operator norm \( \| \cdot \| \) and is denoted \( ℳ(l_1(S)) \) [7]. Taylor shows [6] that \( l_1(S) \) is isometrically isomorphic to a subalgebra of \( M(Γ) \) (norm denoted by \( \| \cdot \| \)), and it is proved in [4] that \( ℳ(l_1(S)) \) is also isomorphic to a subalgebra of \( M(Γ) \) with \( \| \cdot \| \) equivalent to \( \| \cdot \| \) in this case.

3. Multipliers of S. A function \( σ:S→S \) having the property that \( σ(xy) = xσ(y) \) for all \( x, y ∈ S \) is called a multiplier of S. Let \( Ω(S) \) denote the set of all multipliers of S and consider S to be a subsemigroup of \( Ω(S) \). The natural isomorphism \( i \) of S into Γ can be extended to an isomorphism of \( Ω(S) \) into Γ if and only if Γ has an identity, that is, if and only if \( l_1(S) \) has a weak bounded approximate identity of norm one [4]. In this section we show that, by utilizing the relative units which belong to S because \( l_1(S) \) has an approximate identity, then it is possible to imbed in Γ a subsemigroup of \( Ω(S) \) containing S. The importance of this subsemigroup will be seen in the next section. We begin with a technical lemma.

**Lemma 3.1.** Assume that S is imbedded in Γ. For each \( i, 1 ≤ i ≤ n \), there exist nets \( \{x_{p(i)}\}⊂S \) and \( \{u_{p(i)}\}⊂S \) such that

\[
\begin{align*}
(a) & \quad \lim x_{p(i)} = y_i, \quad \text{with } γ_jx_{p(i)} = x_{p(i)} \quad \text{for all } p(i), \\
(b) & \quad u_{p(i)}x_{p(i)} = x_{p(i)}, \quad γ_ju_{p(i)} = u_{p(i)} \quad \text{for all } p(i).
\end{align*}
\]

**Proof.** Let \( i \) be arbitrary, \( 1 ≤ i ≤ n \). Since S is dense in Γ, there exists a net \( \{x_{p'(i)}\}⊂S \) such that \( \lim x_{p'(i)} = y_i \). Then there exists a subnet \( \{x_{p'(i)}\} \) such that \( γ_jx_{p'(i)} = x_{p'(i)} \) for all \( p'(i) \). To see that such a subnet exists observe that there exist \( j \) and a subnet \( \{x_{d(i)}\} \) of \( \{x_{p'(i)}\} \) such that \( γ_jx_{d(i)} = x_{d(i)} \) for all \( d(i) \). But \( γ_jγ_j = \lim γ_jx_{d(i)} = \lim x_{d(i)} = γ_i \) implies that \( U \) is not a minimal set of relative units for Γ unless \( j=i \).

Now, since \( l_1(S) \) has an approximate identity, S has a set of relative units [4], and so there exists a net \( \{u_{p(i)}\}⊂S \) such that \( u_{p(i)}x_{p(i)} = x_{p(i)} \) for all \( p(i) \). Further, let \( \{u_{p(i)}\} \) be a subnet of \( \{u_{p(i)}\} \) such that \( γ_ju_{p(i)} = u_{p(i)} \) for all \( p(i) \). To substantiate the existence of this subnet, once again observe that there exist \( j \) and a subnet \( \{u_{d(i)}\} \) of \( \{u_{p(i)}\} \) such that \( γ_ju_{d(i)} = u_{d(i)} \) for all \( d(i) \). But

\[
γ_jγ_j = \lim γ_jx_{d(i)} = \lim γ_ju_{d(i)}x_{d(i)} = \lim u_{d(i)}x_{d(i)} = \lim x_{d(i)} = γ_i
\]

shows that \( U \) is not a minimal set of relative units for Γ unless \( j=i \). Thus, the nets \( \{x_{p(i)}\} \) and \( \{u_{p(i)}\} \) have the properties stated in the lemma. This completes the proof.
It is possible to use the semicharacters of $T$ to decompose $T$ into the following subsemigroups: $H_T = \{ \nu \in \Gamma: \chi(\gamma) = 0 \text{ if and only if } \chi(\gamma) = 0 \text{ for all } \chi \in \Gamma \}$; $A_T = \{ \nu \in \Gamma: \text{ if } \chi \in \Gamma \text{ and } \chi(\gamma) \neq 0 \text{ then } \chi(\nu) \neq 0 \}$. Also note that the characteristic function of $A_T$ is always a semicharacter of $\Gamma$ [1]. In the next theorem we show that $U$ determines a subset of $\Omega(S)$.

**Theorem 3.2.** Assume that $S$ is imbedded in $\Gamma$ and that nets $\{x_p(i)\} \subset S$, $\{u_p(i)\} \subset S$ are as described in Lemma 3.1, $i=1, 2, \cdots, n$. Then for each $x \in S$ there exists $\rho(i)_x$ such that $\rho(i)_x \geq \rho(i)_x$ implies $xu_p(i) = x\gamma_i$, $i=1, 2, \cdots, n$. Thus, for each $i$, $1 \leq i \leq n$, $\gamma_i \subset S$ and hence multiplication by $\gamma_i$ determines a multiplier of $S$.

**Proof.** Let $x \in S$ and $i$ be arbitrary, $1 \leq i \leq n$. Since the characteristic function $\phi$ of $A_{\gamma_i}$ belongs to $\Gamma$ and $\lim x x_p(i) = x\gamma_i$, then $\phi(x x_p(i)) = \phi(x\gamma_i) = 1$ implies that there exists $\rho(i)_x$ such that $\phi(x_p(i)) = \phi(x) = \phi(x x_p(i)) = \phi(x\gamma_i) = 1$ for all $\rho(i)_x \geq \rho(i)_x$. Hence, by the nature of $\{u_p(i)\}$, $\phi(u_p(i)) = 1$ for all $\rho(i)_x \geq \rho(i)_x$. We thus have that $\rho(i)_x \geq \rho(i)_x$ implies that $x u_p(i) \in A_{\gamma_i}$, $x u_p(i) \in A_{\gamma_i}$, i.e., it is easy to verify that $x u_p(i) \in H_{\gamma_i}$ for all $\rho(i)_x \geq \rho(i)_x$. We assert that $x u_p(i) = x\gamma_i$ for all $\rho(i)_x \geq \rho(i)_x$. If this statement is false, then there exist $\rho(i)_x$ and $\chi \in \Gamma$ that $\chi(\gamma_i) \neq 0$ and $\chi(\gamma_i) \neq 0$. Also, since $x u_p(i) \in A_{\gamma_i}$, then $\chi(x u_p(i)) \neq 0$; and since $x u_p(i) \mu_p(i) = x u_p(i)$, then $\chi(u_p(i)) = 1$. Thus, $\chi(\gamma_i) = u_p(i)$ implies that $\chi(\gamma_i) = 1$; but this leads to the contradiction that $\chi(x) = \chi(x u_p(i)) \neq \chi(x\gamma_i) = \chi(x)$. Therefore, the theorem is proved.

For each $i$, $1 \leq i \leq n$, define a function $\sigma_{\gamma_i}: S \rightarrow S$ by $\sigma_{\gamma_i}(x) = x\gamma_i$ for all $x \in S$, where the product is formed in $\Gamma$. Then $\sigma_{\gamma_i} \in \Omega(S)$ for all $i$. Further, define $D = \{ \sigma_{\gamma_i}, \sigma \in \Omega(S), i=1, 2, \cdots, n \}$; then $D$ is a subsemigroup of $\Omega(S)$.

**Theorem 3.3.** There is an algebra isomorphism $i_D$ of $D$ into $\Gamma$ which extends the natural isomorphism $i_i$ of $S$ into $\Gamma$.

**Proof.** Let $i_i(x) = x$ for all $x \in S$, and let $\sigma, \sigma \in D$, $\sigma \in \Omega(S)$. By Lemma 3.1 there exists a net $\{x_p(i)\} \subset S$ such that $x_p(i) \rightarrow x$. Consider $\{\sigma(x_p(i))\} \subset \Gamma$ and in accordance with the compactness of $\Gamma$ let $\sigma(x_p(i)) = \gamma_i$, $\gamma_i \in \Gamma$. Now, if $x \in S$, then $\gamma_i \gamma_i = \gamma_i \sigma(x_p(i)) = \gamma_i \sigma(x_p(i)) = \gamma_i \gamma_i$. Suppose that $\{x_p(i)\} \subset S$ is any net such that $\lim x_p(i) = \gamma_i$, and further suppose that $\{\sigma(x_p(i))\}$ is a convergent subnet of $\{\sigma(x_p(i))\}$ with $\lim \sigma(x_p(i)) = \gamma_i \gamma_i$, $\gamma_i \gamma_i \in \Gamma$. Now, if $x \in S$, then $\gamma_i \gamma_i = \lim \sigma(x_p(i)) = \lim \sigma(x) \gamma_i \gamma_i = \sigma(x) \gamma_i \gamma_i$. Therefore, the theorem is proved.
When $\Gamma$ has an identity, $D = \Omega(S)$ and the isomorphism $i_D$ is simply an imbedding of $\Omega(S)$ into $\Gamma$; $i_D$ then induces an isomorphism of $l_1(\Omega(S))$ into $M(\Gamma)$.

4. Multipliers of $l_1(S)$ induced by $l_1(\Omega(S))$. In this paragraph and in Proposition 4.1 we drop the restriction that $l_1(S)$ has an approximate identity. Since $S \subseteq \Omega(S)$, $l_1(S)$ can be viewed as a subalgebra of $l_1(\Omega(S))$. Let $l_1(S)^0$ denote the annihilator of $l_1(S)$ in $l_1(\Omega(S))$, that is $l_1(S)^0 = \{ \tau \in l_1(\Omega(S)) : \tau \ast \alpha = 0 \text{ for all } \alpha \in l_1(S) \}$. It follows that $l_1(S)^0$ is a closed ideal in $l_1(\Omega(S))$, in which case $l_1(\Omega(S))/l_1(S)^0$ is a Banach algebra under quotient norm $\| \cdot \|_Q$. Now, there is a natural homomorphism $\tau \mapsto T_\tau$ of $l_1(\Omega(S))$ into $M(l_1(S))$ given by $T_\tau(\alpha) = \tau \ast \alpha$ for all $\alpha \in l_1(S)$. However, in general this homomorphism cannot be expected to be one-to-one. But the following is true.

**Proposition 4.1.** (1) The natural map of $l_1(S)$ into $l_1(\Omega(S))/l_1(S)^0$ is one-to-one and $l_1(\Omega(S))/l_1(S)^0$ is semisimple.

(2) The induced homomorphism $\tau + l_1(S)^0 \mapsto T_\tau$ of $l_1(\Omega(S))/l_1(S)^0$ into $M(l_1(S))$ is one-to-one and $\| T_\tau \| = \| T + l_1(S)^0 \|_Q$.

**Proof.** (1) If $\alpha + l_1(S)^0 = l_1(S)^0$ for some $\alpha \in l_1(S)$, then $\alpha \ast \delta_x = 0$ for all $x \in S$, in contradiction to the semisimplicity of $l_1(S)$ unless $\alpha = 0$.

Also if $\tau + l_1(S)^0 \neq l_1(S)^0$, then choose $\alpha \in l_1(S)$ such that $0 \neq \tau \ast \alpha \in l_1(S)$. Semisimplicity of $l_1(S)$ implies the existence of $\chi \in \Gamma$ such that $(\tau \ast \alpha) \delta_\chi \neq 0$, which in turn implies $\hat{\tau}(\chi) \neq 0$.

(2) To see that $\tau + l_1(S)^0 \mapsto T_\tau$ is a one-to-one map, observe that $T_\tau = 0$ implies that $\tau \ast \alpha = T_\tau(\alpha) = 0$ for all $\alpha \in l_1(S)$, in which case $\tau \in l_1(S)^0$.

If $\tau \in l_1(\Omega(S))$, $\tau' \in l_1(S)^0$, then for all $\alpha \in l_1(S)$ we have $\| \tau \ast \alpha \| = \| \tau \ast \alpha + \tau' \ast \alpha \| \leq \| \tau + \tau' \| \| \alpha \| ;$ thus, $\| T_\tau \| \leq \| T + l_1(S)^0 \|_Q$. This completes the proof.

If $l_1(S)$ has a bounded approximate identity that is not of norm one, nonzero annihilators of $l_1(S)$ are quite readily available. If $e$ denotes the identity of $\Omega(S)$ and

$$ E = \sum_{1 \leq i \leq n} \delta_{x_i} - \sum_{1 \leq i < j \leq n} \delta_{x_i y_j} + \cdots + (-1)^{n+1} \delta_{x_1 y_1 \cdots y_n}, $$

then $(e - E) \in l_1(S)^0$ and so, for every $\tau \in l_1(\Omega(S))$, $(\tau - \tau \ast E) \in l_1(S)^0$.

On the other hand, if $l_1(S)$ has a bounded approximate identity of norm one, then because $M(l_1(S))$ is isometrically isomorphic to $l_1(\Omega(S))$ [4], $l_1(S)^0 = 0$ according to Proposition 4.1.

Assume for the remainder of this paper that $l_1(S)$ has a bounded approxi-

mate identity and that $E$ is defined as above. Consider

$$ E \ast l_1(\Omega(S)) = \{ E \ast \tau : \tau \in l_1(\Omega(S)) \}; $$
it is straightforward to show that $E \ast l_1(\Omega(S))$ is a closed subalgebra of $l_1(\Omega(S))$ and hence complete. Moreover, we are now in a position to prove the main result, namely that $\mathcal{M}(l_1(S))$ is topologically isomorphic to $l_1(\Omega(S))/l_1(S)^0$.

**Theorem 4.2.** (1) $E \ast l_1(\Omega(S))$ is isometrically isomorphic to $l_1(D)\subset l_1(\Gamma)$.

(2) $l_1(\Omega(S))/l_1(S)^0$ is isomorphic to $E \ast l_1(\Omega(S))$, and hence $\| \cdot \|_Q$ is equivalent to $\| \cdot \|$. Moreover, $l_1(\Omega(S))/l_1(S)^0$ is an $l_1$-algebra.

(3) $\mathcal{M}(l_1(S))$ is isomorphic to $l_1(\Omega(S))/l_1(S)^0$ and $\| \cdot \|$ is equivalent to $\| \cdot \|_Q$.

**Proof.** Statements (1) and (2) are fairly obvious.

(1) Using the notation of Theorem 3.3, $i_D:D\to \Gamma$ is defined by $i_D(\sigma_1,\ldots,\sigma_n)=\sigma_i$, for all $\sigma \in \Omega(S)$, $i=1,\ldots,n$. The isomorphism $i_D$ induces a mapping from $E \ast l_1(\Omega(S))$ into $l_1(\Gamma)$ by defining, for all $\sigma \in \Omega(S)$,

$$E \ast \delta_\sigma \mapsto \sum_{1 \leq i \leq n} \delta_{i,\sigma} - \sum_{1 \leq i < j \leq n} \delta_{i,\sigma_j,\sigma} \cdot \sum_{1 \leq i < j < k \leq n} \delta_{i,\sigma_j,\sigma_k,\sigma} + \cdots + (-1)^{n+1} \delta_{1,\sigma_2,\ldots,\sigma_n,\sigma}.$$  

Identifying $D$ with $i_D(D)$ this induced homomorphism is clearly an isometry of $E \ast l_1(\Omega(S))$ onto $l_1(D)$.

(2) Define a homomorphism from $l_1(\Omega(S))/l_1(S)^0$ to $E \ast l_1(\Omega(S))$ by $\tau+i_D(S)^0 \mapsto E \ast \tau$. This mapping is clearly one-to-one and onto. Thus, because $l_1(\Omega(S))/l_1(S)^0$ is a semisimple Banach algebra complete under $\| \cdot \|$ and $\| \cdot \|_Q$, then the two norms must be equivalent [5]. The fact that $l_1(\Omega(S))/l_1(S)^0$ is an $l_1$-algebra follows from (1).

(3) By Theorem 3.1 of [4], $\mathcal{M}(l_1(S))$ is isomorphic to a subalgebra of $\mathcal{M}(\Gamma)$; we now have that $l_1(\Omega(S))/l_1(S)^0$ is topologically isomorphic to $l_1(D)$, a closed subalgebra of $l_1(\Gamma)$, where $D$ is a subsemigroup of $\Gamma$. If $\Lambda$ denotes the uniformly closed subspace of $l_\infty(S)$ generated by $\tilde{S}$, then the measure $\mu \in \mathcal{M}(\Gamma)$ is a multiplier of $l_1(S)$ if and only if $m_\mu$ is continuous on $\Lambda$ in the weak topology generated by $l_1(S)$, where, $m_\mu:\Lambda\to \Lambda$ is such that $m_\mu(f)$ is the Arens product of $\mu \in \mathcal{M}(\Gamma)$ and $f \in \Lambda$ [4, Definition 3.3 and Corollary 3.8]. In order to prove that $\mathcal{M}(l_1(S))$ is isomorphic to $l_1(D)$, we need only show that if $\mu \in \mathcal{M}(\Gamma)$ is such that $m_\mu$ is continuous on $\Lambda$, then $\mu$ determines a multiplier of $l_1(D)$ and hence belongs to $l_1(D)$ since $l_1(D)$ has an identity. Now, because $D$ is a subset of $\Gamma$, $\Lambda$ can be identified with a subset of $l_1(D)^*$ as in [4, Proposition 4.8: replace $\Omega(S)$ by $D$]. Likewise, Theorem 4.9 of [4] establishes the desired isomorphism between $\mathcal{M}(l_1(S))$ and $l_1(D)$ by replacing $\Omega(S)$ by $D$ throughout that part of the proof. Theorem 3.1 of [4] and (2) then prove that $\| \cdot \|$ is equivalent to $\| \cdot \|_Q$.  

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EXAMPLE 4.3. Let $S$ be the set of integers under maximum multiplication and consider the subgroup $S_0$ of $S \times S$ consisting of the negative axes $\{(x, 0), (0, y) : x, y \in S, x \leq 0, y \leq 0\}$. $\Gamma$ contains relative units $\gamma_1$ and $\gamma_2$ with products: $\gamma_1 \gamma_2 = (0, 0)$; $\gamma_1(n, 0) = (n, 0)$ for all $n \leq 0$; $\gamma_1(0, m) = (0, 0)$ for all $m \leq 0$; $\gamma_2(0, m) = (0, m)$ for all $m \leq 0$; and $\gamma_2(n, 0) = (0, 0)$ for all $n \leq 0$. If $\sigma \in \Omega(S_0)$, then $\sigma$ restricted to $X_1 = \{(x, 0) : x \in S, x \leq 0\}$ is a multiplier of $X_1$: suppose $x \in X_1$ is such that $\sigma(x) \notin X_1$; then $\sigma \gamma_1 = \sigma \gamma_2$ implies that $\sigma(x) = (0, 0) \in X_1$ and hence a contradiction. Similarly, $\sigma$ restricted to $X_2 = \{(0, x) : x \in S, x \leq 0\}$ is a multiplier of $X_2$. Thus, each $\sigma \in \Omega(S_0)$ is of the form $(\sigma', \sigma'')$ where $\sigma' \in \Omega(X_1)$ and $\sigma'' \in \Omega(X_2)$ and

$$(\sigma', \sigma'')(x, 0) = (\sigma'(x), 0), (\sigma', \sigma'')(0, x) = (0, \sigma''(x))$$

for all $x \in S, x \leq 0$.

Also, $(\sigma', \sigma')(v', v'') = (\sigma' v', \sigma'' v'')$ for all $\sigma', \sigma'' \in \Omega(X_1)$, $\sigma'' \in \Omega(X_2)$.

Now because $\Omega(X_i)$ is just $X_i$ with an identity adjoined for $i = 1, 2$, then

$$\Omega(S_0) = \{(\sigma_{y_1}, \sigma_{y_2}) \cup \{(\sigma_{y_1}, \sigma_{y_2}) : x \in S, x \leq 0\}
\cup \{(\sigma_{y_1}, \sigma_{y_2}) : x \in S, x \leq 0\}
\cup \{(\sigma_{y_1}, \sigma_{y_2}) : y \in S, x \leq 0, y \leq 0\}.

However, only the subsemigroup $\{(\sigma_{x_1}, \sigma_0) \cup \{(\sigma_0, \sigma_{y_2}) \cup S_0$ is embedded in $\Gamma$. For a fuller discussion of $S_0$ and $\Gamma$ see [3, Example 4.14].

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