

THE \mathcal{C} -BOREL TRANSFORM

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ABSTRACT. For \mathcal{C} , a given entire function, it is established that the \mathcal{C} -Borel transform is a linear isomorphism of the space dual to a space of admissible holomorphic functions on a disk in the complex plane C onto the space of admissible entire functions of certain growth. The theory is extended to C^n and shown to include the Fourier-Borel and Hankel-Borel transforms as special cases.

1. Introduction. Let \mathcal{C} be a given entire function with the series representation $\mathcal{C}(z) = \sum_{k=0}^{\infty} c_k z^k$, c_k real. For the disk $s(\tau) = \{z \mid |z| < \tau\}$ in the complex plane C , let $\mathcal{H}_{\mathcal{C}}(s(\tau))$ be the Fréchet space of holomorphic functions g on $s(\tau)$ such that $g(z) = \sum_{k=0}^{\infty} g_k z^k$ in the neighborhood of the origin and $\text{supp}(g_n) \subset \text{supp}(c_n)$. It is our goal to establish that the elements of the dual space $\mathcal{H}'_{\mathcal{C}}(s(\tau))$ can be given a natural interpretation as entire functions of a certain rate of growth. The results will then be extended to C^n . The theory developed includes as special cases that of the classical Fourier-Borel transform, [3, Chapter 22], and of the Hankel-Borel transform [1].

2. Notations and definitions. Let \mathcal{C} be a given entire function represented by

$$(1) \quad \mathcal{C}(z) = \sum_{k=0}^{\infty} c_k z^k, \quad c_k \text{ real,}$$

and denote by \mathcal{C}^* the function given by

$$(2) \quad \mathcal{C}^*(z) = \sum_{k=0}^{\infty} |c_k| z^k.$$

DEFINITION 1. A function f is said to be admissible if it can be represented by a series

$$(3) \quad f(z) = \sum_{k=0}^{\infty} f_k z^k$$

with

$$(4) \quad \text{supp}(f_k) \subset \text{supp}(c_k).$$

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DEFINITION 2. An admissible entire function f is said to belong to the class $\{\mathcal{C}, \tau\}$ if, for some $\tau > 0$, there exists a constant $M > 0$ such that

$$(5) \quad |f(z)| \leq M\mathcal{C}^*(\tau|z|).$$

We denote by $\{\mathcal{C}, \tau\}^\sim$ the class

$$(6) \quad \{\mathcal{C}, \tau\}^\sim = \bigcup_{\tau' < \tau} \{\mathcal{C}, \tau'\}.$$

DEFINITION 3. An admissible entire function f is said to be of \mathcal{C} -type τ if

$$(7) \quad \tau = \inf\{\sigma \mid |f(z)| \leq M\mathcal{C}^*(\sigma|z|)\}.$$

By an appeal to Nachbin's theorem [2], it can readily be established that an admissible entire function f is of \mathcal{C} -type τ if and only if

$$(8) \quad \limsup_{n \in \text{supp}(c_k)} \left| \frac{f_n}{c_n} \right|^{1/n} = \tau.$$

We denote by $\mathcal{H}_\varphi(\Omega)$ the Fréchet space of admissible holomorphic functions on an open set Ω in C containing the origin. It is immediate that the vector space of admissible polynomials on Ω , $\mathcal{P}(\Omega)$, is a subspace of $\mathcal{H}_\varphi(\Omega)$.

We denote by $\mathcal{H}'_\varphi(\Omega)$ the dual of $\mathcal{H}_\varphi(\Omega)$.

DEFINITION 4. The elements of $\mathcal{H}'_\varphi(\Omega)$ are the \mathcal{C} -analytic functionals in Ω .

3. **Principal theorems in C .** We now establish our principal result.

THEOREM 1. Let τ be a positive number. For any $f \in \{\mathcal{C}, \tau\}^\sim$, the linear functional on $\mathcal{P}(C)$ defined by

$$(9) \quad \langle f, P \rangle = \sum_{\text{supp}(p_k)} \frac{f_k}{c_k} p_k$$

where $P(z) = \sum_{\text{supp}(p_k)} p_k z^k \in \mathcal{P}(C)$ can be extended uniquely to a continuous linear functional σ_f on $\mathcal{H}_\varphi(s(\tau))$. The mapping

$$(10) \quad f \rightarrow \sigma_f$$

is a vector space isomorphism of $\{\mathcal{C}, \tau\}^\sim$ onto $\mathcal{H}'_\varphi(s(\tau))$. The inverse mapping is given by the equation

$$(11) \quad f(t) = \langle \sigma_f, \mathcal{C}(tz) \rangle,$$

where σ_f operates on functions of $z \in s(\tau)$.

PROOF. Since $f \in \{\mathcal{C}, \tau\}^\sim$, we can find a $\tau' < \tau$ and a constant $M > 0$ so that, by virtue of (8), $|f_k/c_k| \leq M\tau'^k$, $k \in \text{supp}(c_n)$. Further, by Cauchy's inequality, we have, for arbitrary $g \in \mathcal{H}_\varphi(C)$,

$$\left| \frac{g^{(k)}(0)}{k!} \right| \leq \frac{1}{\tau''^k} \sup_{|z|=\tau''} |g(z)|, \quad \tau' < \tau'' < \tau.$$

Hence, for any $p \in \mathcal{P}(C)$ with $P(z) = \sum_{\text{supp}(p_k)} p_k z^k$, we have that

$$\begin{aligned} |\langle f, P \rangle| &= \left| \sum_{\text{supp}(p_k)} \frac{f_k}{c_k} p_k \right| \\ &\leq M \sum_{\text{supp}(p_k)} \left(\frac{\tau'}{\tau''} \right)^k \sup_{|z|=\tau''} |P(z)| \leq \frac{M}{1 - \tau'/\tau''} \sup_{|z|=\tau''} |P(z)|. \end{aligned}$$

It follows that the linear functional is continuous on $\mathcal{P}(C)$ with the topology induced by $\mathcal{H}_\varphi(s(\tau))$ and so has a unique extension to a \mathcal{C} -analytic functional σ_f on $\mathcal{H}_\varphi(s(\tau))$, since the set of admissible polynomials is dense in $\mathcal{H}_\varphi(s(\tau))$.

We have that $f \rightarrow \sigma_f$ is a linear map of $\{\mathcal{C}, \tau\}^\sim$ into $\mathcal{H}'_\varphi(s(\tau))$.

To show that the map is one-to-one, it suffices to establish that $\sigma_f = 0$ if and only if $f = 0$. But, for $P(z) = z^n$, $n \in \text{supp}(c_k)$, we have $\langle \sigma_f, P \rangle = f_n/c_n \cdot 1 = 0$ if and only if $f_n = 0$, and the result follows.

To show that the map is onto, let σ be an arbitrary \mathcal{C} -analytic functional in $s(\tau)$. Since $\mathcal{C}(tz)$ is an entire function, its restriction to $s(\tau)$ clearly belongs to $\mathcal{H}_\varphi(s(\tau))$. It thus follows that

$$(12) \quad \langle \sigma, \mathcal{C}(tz) \rangle$$

is a well-defined function of $t \in C$, and we denote this function by f .

We now establish that $f \in \{\mathcal{C}, \tau\}^\sim$. As a consequence of the continuity of σ , we have that there is a constant $M > 0$ and a compact subset K of $s(\tau)$ such that, for all $g \in \mathcal{H}_\varphi(s(\tau))$, $|\langle \sigma, g \rangle| \leq M \sup_{z \in K} |g(z)|$. Now, for $K \subset s(\tau')$, $\tau' < \tau$, and taking $g(z) = z^k$, $k \in \text{supp}(c_n)$, we have

$$(13) \quad |\langle \sigma, z^k \rangle| \leq M\tau'^k.$$

On the other hand, again because of the continuity of σ , we have that

$$(14) \quad f(t) = \langle \sigma, \mathcal{C}(tz) \rangle = \sum_{\text{supp}(c_k)} c_k \langle \sigma, z^k \rangle t^k$$

with the series on the right converging for all $t \in C$ by (13). Hence f is an admissible entire function. Further, $f \in \{\mathcal{C}, \tau\}^\sim$ since by (13),

$$|f(t)| \leq M \sum_{\text{supp}(c_k)} |c_k| (\tau' |t|)^k \leq M \mathcal{C}^*(\tau' |t|).$$

Finally, we have that $\sigma_f = \sigma$; for, if $P \in \mathcal{P}(C)$, then taking note of (14), we have

$$\begin{aligned} \langle \sigma_f, P \rangle &= \sum_{\text{supp}(p_k)} \frac{f_k}{c_k} p_k = \sum_{\text{supp}(p_k)} \langle \sigma, z^k \rangle p_k \\ &= \left\langle \sigma, \sum_{\text{supp}(p_k)} p_k z^k \right\rangle = \langle \sigma, P \rangle. \end{aligned}$$

Since $\mathcal{P}(C)$ is dense in $\mathcal{H}_\varphi(s(\tau))$, the proof is complete.

With the introduction of the following definitions, Theorem 1 may be restated.

DEFINITION 5. A \mathcal{C} -analytic functional σ in C is said to be carried by an open set $\Omega \subset C$ if there is a compact subset K of Ω and a constant $M > 0$ such that for all admissible entire functions g in C ,

$$(15) \quad |\langle \sigma, g \rangle| \leq M \sup_{z \in K} |g(z)|.$$

We note that σ is carried by Ω if the linear form $g \rightarrow \langle \sigma, g \rangle$ defined on the restriction to Ω of the admissible entire functions g can be extended as a continuous linear form to the whole of $\mathcal{H}_\varphi(\Omega)$.

DEFINITION 6. If σ is a \mathcal{C} -analytic functional in C , the function of $t \in C$,

$$(16) \quad \langle \sigma, \mathcal{C}(tz) \rangle$$

is the \mathcal{C} -Borel transform of σ and is denoted by $\hat{\sigma}$.

The preceding theorem can now be expressed in the following form.

THEOREM 2. *The \mathcal{C} -Borel transform is a linear isomorphism of $\mathcal{H}'_\varphi(C)$ onto the space of admissible entire functions of \mathcal{C} -type on C . Further, the \mathcal{C} -analytic functional $\sigma \in \mathcal{H}'_\varphi(C)$ is carried by the open disk $s(\tau)$ if and only if there exists a τ' , $0 < \tau' < \tau$, such that $\hat{\sigma} \in \{\mathcal{C}, \tau'\}$.*

4. Extension to C^n . The preceding theory may be generalized by extension to higher dimensions. To this end, let

$$(17) \quad \mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \dots, \mathcal{C}^{(n)}$$

be n entire functions of the form (1) and

$$(18) \quad \mathcal{C}^{(1)*}, \mathcal{C}^{(2)*}, \dots, \mathcal{C}^{(n)*}$$

the functions corresponding to (2). For

$$(19) \quad Z = (z_1, \dots, z_n) \in C^n,$$

define

$$(20) \quad \begin{aligned} \mathcal{C}(Z) &= \mathcal{C}^{(1)}(z_1)\mathcal{C}^{(2)}(z_2)\cdots\mathcal{C}^{(n)}(z_n) \\ &= \sum_{\alpha \in \mathbb{N}^n} c_\alpha Z^\alpha = \sum_{\alpha_k \in \mathbb{N}} c_{\alpha_1} c_{\alpha_2} \cdots c_{\alpha_n} z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}. \end{aligned}$$

DEFINITION 7. A function f on C^n is admissible if

$$(21) \quad f(Z) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha Z^\alpha$$

where

$$(22) \quad \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$$

with

$$(23) \quad \text{supp}(f_\alpha) \subset \text{supp}(c_\alpha)$$

and

$$(24) \quad Z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}.$$

DEFINITION 8. An admissible entire function f belongs to class $\{\mathcal{C}, T\}$ if, for some $T=(\tau_1, \tau_2, \cdots, \tau_n)$, $\tau_k > 0$, $k=1, \cdots, n$, there exists a constant $M > 0$ such that

$$(25) \quad |f(Z)| \leq M \mathcal{C}^{(1)*}(\tau_1 |z_1|) \mathcal{C}^{(2)*}(\tau_2 |z_2|) \cdots \mathcal{C}^{(n)*}(\tau_n |z_n|).$$

We denote by $\{\mathcal{C}, T\}^\sim$ the union of classes $\{\mathcal{C}, T'\}$, $\tau'_k < \tau_k$, $k=1, \cdots, n$. Analogous to the case in the complex plane, we have that if $f \in \{\mathcal{C}, T\}^\sim$, then

$$(26) \quad |f_\alpha/c_\alpha| \leq MT'^\alpha$$

for all α satisfying (23).

We let $\mathcal{H}_\varphi(s(T))$ be the space of admissible holomorphic functions on the polydisk $s(T)$.

The extension of Theorem 1 to C^n can then be stated as follows.

THEOREM 3. Let $T=(\tau_1, \cdots, \tau_n)$, $\tau_k > 0$, $k=1, \cdots, n$. For any $f \in \{\mathcal{C}, T\}^\sim$, the linear functional on the space $\mathcal{P}(C^n)$ of polynomials in n complex variables defined by

$$(27) \quad \langle f, P \rangle = \sum_{\alpha \in \text{supp}(p_\alpha)} \frac{f_\alpha}{c_\alpha} p_\alpha,$$

where $P(z) = \sum_{\alpha \in \text{supp}(p_\alpha)} p_\alpha Z^\alpha \in \mathcal{P}(C^n)$, can be extended uniquely to a continuous linear functional σ_f on $\mathcal{H}_\varphi(s(T))$. Further, the mapping

$$(28) \quad f \rightarrow \sigma_f$$

is a vector space isomorphism of $\{\mathcal{C}, T\} \sim$ onto $\mathcal{H}'_{\mathcal{C}}(s(T))$. The inverse mapping is given by the equation

$$(29) \quad f(T) = \langle \sigma_f, \mathcal{C}(W) \rangle,$$

$T = (t_1, \dots, t_n)$, $W = (t_1 z_1, t_2 z_2, \dots, t_n z_n)$, where σ_f operates on functions of $Z \in s(T)$.

5. **Applications.** When each of the functions of (17) is the exponential function in the complex plane, we have that

$$(30) \quad \mathcal{C}(Z) = e^{z_1} e^{z_2} \cdots e^{z_n} = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} Z^\alpha$$

and it is clear, with $c_\alpha = 1/\alpha!$, that all entire functions are admissible. In this case the continuous linear functionals on the space of holomorphic functions on a polydisk may be interpreted as entire functions of exponential type. Further, since $\mathcal{C}(W) = e^{\langle Z, T \rangle}$,

$$\langle Z, T \rangle = z_1 t_1 + z_2 t_2 + \cdots + z_n t_n,$$

the \mathcal{C} -Borel transform (29) can be written in the form

$$\langle \sigma_f, e^{\langle Z, T \rangle} \rangle$$

which is the classical Fourier-Borel transform.

If each of the functions of (17) is equal to

$$\mathcal{J}(z) = 2^{\nu-1/2} \Gamma(\nu + \frac{1}{2}) z^{1/2-\nu} J_{\nu-1/2}(z),$$

J_α being the Bessel function of order α , then if

$$b_{2k} = \frac{(-1)^k \Gamma(\nu + \frac{1}{2})}{2^{2k} k! \Gamma(\nu + \frac{1}{2} + k)},$$

we have that

$$\mathcal{C}(Z) = \mathcal{J}(z_1) \mathcal{J}(z_2) \cdots \mathcal{J}(z_n) = \sum_{\alpha_k \text{ even}} b_{\alpha_1} b_{\alpha_2} \cdots b_{\alpha_n} z_1^{\alpha_1} \cdots z_n^{\alpha_n},$$

and it follows that all even entire functions are admissible. Here the linear functionals on the space of even holomorphic functions on a polydisk may be identified with the even entire functions of certain growth. The transform (29) in this case becomes the Hankel-Borel transform $\langle \sigma_f, \prod_{k=1}^n \mathcal{J}(z_k t_k) \rangle$.

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