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THE $\mathscr{C}$-BOREL TRANSFORM

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ABSTRACT. For $\mathscr{C}$, a given entire function, it is established that the $\mathscr{C}$-Borel transform is a linear isomorphism of the space dual to a space of admissible holomorphic functions on a disk in the complex plane C onto the space of admissible entire functions of certain growth. The theory is extended to C^n and shown to include the Fourier-Borel and Hankel-Borel transforms as special cases.

1. **Introduction.** Let $\mathscr{C}$ be a given entire function with the series representation $\mathscr{C}(z)=\sum_{k=0}^{\infty} c_k z^k$, c_k real. For the disk $s(\tau)=\{z|\,|z|<\tau\}$ in the complex plane C, let $\mathscr{H}_{\mathscr{C}}(s(\tau))$ be the Fréchet space of holomorphic functions g on $s(\tau)$ such that $g(z)=\sum_{k=0}^{\infty} g_k z^k$ in the neighborhood of the origin and $\operatorname{supp}(g_n)\subset \operatorname{supp}(c_n)$. It is our goal to establish that the elements of the dual space $\mathscr{H}'_{\mathscr{C}}(s(\tau))$ can be given a natural interpretation as entire functions of a certain rate of growth. The results will then be extended to C^n. The theory developed includes as special cases that of the classical Fourier-Borel transform, [3, Chapter 22], and of the Hankel-Borel transform [1].

2. **Notations and definitions.** Let $\mathscr{C}$ be a given entire function represented by

(1) $$\mathscr{C}(z) = \sum_{k=0}^{\infty} c_k z^k, \qquad c_k \text{ real},$$

and denote by $\mathscr{C}^*$ the function given by

(2) $$\mathscr{C}^*(z) = \sum_{k=0}^{\infty} |c_k|\, z^k.$$

DEFINITION 1. A function f is said to be admissible if it can be represented by a series

(3) $$f(z) = \sum_{k=0}^{\infty} f_k z^k$$

with

(4) $$\operatorname{supp}(f_k) \subset \operatorname{supp}(c_k).$$

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DEFINITION 2. An admissible entire function f is said to belong to the class $\{\mathscr{C}, \tau\}$ if, for some $\tau>0$, there exists a constant $M>0$ such that

$$|f(z)| \leqq M\mathscr{C}^*(\tau\,|z|). \tag{5}$$

We denote by $\{\mathscr{C}, \tau\}^\sim$ the class

$$\{\mathscr{C}, \tau\}^\sim = \bigcup_{\tau'<\tau} \{\mathscr{C}, \tau'\}. \tag{6}$$

DEFINITION 3. An admissible entire function f is said to be of $\mathscr{C}$-type τ if

$$\tau = \inf\{\sigma \mid |f(z)| \leqq M\mathscr{C}^*(\sigma\,|z|)\}. \tag{7}$$

By an appeal to Nachbin's theorem **[2]**, it can readily be established that an admissible entire function f is of $\mathscr{C}$-type τ if and only if

$$\limsup_{n\in\mathrm{supp}(c_k)} \left|\frac{f_n}{c_n}\right|^{1/n} = \tau. \tag{8}$$

We denote by $\mathscr{H}_{\mathscr{C}}(\Omega)$ the Fréchet space of admissible holomorphic functions on an open set Ω in C containing the origin. It is immediate that the vector space of admissible polynomials on Ω, $\mathscr{P}(\Omega)$, is a subspace of $\mathscr{H}_{\mathscr{C}}(\Omega)$.

We denote by $\mathscr{H}'_{\mathscr{C}}(\Omega)$ the dual of $\mathscr{H}_{\mathscr{C}}(\Omega)$.

DEFINITION 4. The elements of $\mathscr{H}'_{\mathscr{C}}(\Omega)$ are the $\mathscr{C}$-analytic functionals in Ω.

3. **Principal theorems in** C. We now establish our principal result.

THEOREM 1. *Let τ be a positive number. For any $f\in\{\mathscr{C}, \tau\}^\sim$, the linear functional on $\mathscr{P}(C)$ defined by*

$$\langle f, P\rangle = \sum_{\mathrm{supp}(p_k)} \frac{f_k}{c_k}\, p_k \tag{9}$$

where $P(z)=\sum_{\mathrm{supp}(p_k)} p_k z^k \in \mathscr{P}(C)$ can be extended uniquely to a continuous linear functional σ_f on $\mathscr{H}_{\mathscr{C}}(s(\tau))$. The mapping

$$f \to \sigma_f \tag{10}$$

is a vector space isomorphism of $\{\mathscr{C}, \tau\}^\sim$ onto $\mathscr{H}'_{\mathscr{C}}(s(\tau))$. The inverse mapping is given by the equation

$$f(t) = \langle \sigma_f, \mathscr{C}(tz)\rangle, \tag{11}$$

where σ_f operates on functions of $z\in s(\tau)$.

PROOF. Since $f \in \{\mathscr{C}, \tau\}^{\sim}$, we can find a $\tau' < \tau$ and a constant $M > 0$ so that, by virtue of (8), $|f_k/c_k| \leqq M\tau'^k$, $k \in \operatorname{supp}(c_n)$. Further, by Cauchy's inequality, we have, for arbitrary $g \in \mathscr{H}_{\mathscr{C}}(C)$,

$$\left|\frac{g^{(k)}(0)}{k!}\right| \leqq \frac{1}{\tau''^k} \sup_{|z|=\tau''} |g(z)|, \qquad \tau' < \tau'' < \tau.$$

Hence, for any $p \in \mathscr{P}(C)$ with $P(z) = \sum_{\operatorname{supp}(p_k)} p_k z^k$, we have that

$$\begin{aligned}|\langle f, P\rangle| &= \left|\sum_{\operatorname{supp}(p_k)} \frac{f_k}{c_k} p_k\right| \\ &\leqq M \sum_{\operatorname{supp}(p_k)} \left(\frac{\tau'}{\tau''}\right)^k \sup_{|z|=\tau''} |P(z)| \leqq \frac{M}{1 - \tau'/\tau''} \sup_{|z|=\tau''} |P(z)|.\end{aligned}$$

It follows that the linear functional is continuous on $\mathscr{P}(C)$ with the topology induced by $\mathscr{H}_{\mathscr{C}}(s(\tau))$ and so has a unique extension to a $\mathscr{C}$-analytic functional σ_f on $\mathscr{H}_{\mathscr{C}}(s(\tau))$, since the set of admissible polynomials is dense in $\mathscr{H}_{\mathscr{C}}(s(\tau))$.

We have that $f \to \sigma_f$ is a linear map of $\{\mathscr{C}, \tau\}^{\sim}$ into $\mathscr{H}'_{\mathscr{C}}(s(\tau))$.

To show that the map is one-to-one, it suffices to establish that $\sigma_f = 0$ if and only if $f = 0$. But, for $P(z) = z^n$, $n \in \operatorname{sup}(c_k)$, we have $\langle \sigma_f, P\rangle = f_n/c_n \cdot 1 = 0$ if and only if $f_n = 0$, and the result follows.

To show that the map is onto, let σ be an arbitrary $\mathscr{C}$-analytic functional in $s(\tau)$. Since $\mathscr{C}(tz)$ is an entire function, its restriction to $s(\tau)$ clearly belongs to $\mathscr{H}_{\mathscr{C}}(s(\tau))$. It thus follows that

$$\langle \sigma, \mathscr{C}(tz)\rangle \tag{12}$$

is a well-defined function of $t \in C$, and we denote this function by f.

We now establish that $f \in \{\mathscr{C}, \tau\}^{\sim}$. As a consequence of the continuity of σ, we have that there is a constant $M > 0$ and a compact subset K of $s(\tau)$ such that, for all $g \in \mathscr{H}_{\mathscr{C}}(s(\tau))$, $|\langle \sigma, g\rangle| \leqq M \sup_{z \in K} |g(z)|$. Now, for $K \subset s(\tau')$, $\tau' < \tau$, and taking $g(z) = z^k$, $k \in \operatorname{supp}(c_n)$, we have

$$|\langle \sigma, z^k\rangle| \leqq M\tau'^k. \tag{13}$$

On the other hand, again because of the continuity of σ, we have that

$$f(t) = \langle \sigma, \mathscr{C}(tz)\rangle = \sum_{\operatorname{supp}(c_k)} c_k \langle \sigma, z^k\rangle t^k \tag{14}$$

with the series on the right converging for all $t \in C$ by (13). Hence f is an admissible entire function. Further, $f \in \{\mathscr{C}, \tau\}^{\sim}$ since by (13),

$$|f(t)| \leqq M \sum_{\operatorname{supp}(c_k)} |c_k| (\tau' |t|)^k \leqq M\mathscr{C}^*(\tau' |t|).$$

Finally, we have that $\sigma_f=\sigma$; for, if $P\in\mathscr{P}(C)$, then taking note of (14), we have

$$\langle\sigma_f, P\rangle = \sum_{\operatorname{supp}(p_k)} \frac{f_k}{c_k} p_k = \sum_{\operatorname{supp}(p_k)} \langle\sigma, z^k\rangle p_k$$

$$= \left\langle \sigma, \sum_{\operatorname{supp}(p_k)} p_k z^k \right\rangle = \langle\sigma, P\rangle.$$

Since $\mathscr{P}(C)$ is dense in $\mathscr{H}_{\mathscr{C}}(s(\tau))$, the proof is complete.

With the introduction of the following definitions, Theorem 1 may be restated.

DEFINITION 5. A $\mathscr{C}$-analytic functional σ in C is said to be carried by an open set $\Omega\subset C$ if there is a compact subset K of Ω and a constant $M>0$ such that for all admissible entire functions g in C,

$$|\langle\sigma, g\rangle| \leqq M \sup_{z\in K} |g(z)|. \tag{15}$$

We note that σ is carried by Ω if the linear form $g\to\langle\sigma, g\rangle$ defined on the restriction to Ω of the admissible entire functions g can be extended as a continuous linear form to the whole of $\mathscr{H}_{\mathscr{C}}(\Omega)$.

DEFINITION 6. If σ is a $\mathscr{C}$-analytic functional in C, the function of $t\in C$,

$$\langle\sigma, \mathscr{C}(tz)\rangle \tag{16}$$

is the $\mathscr{C}$-Borel transform of σ and is denoted by $\hat{\sigma}$.

The preceding theorem can now be expressed in the following form.

THEOREM 2. *The $\mathscr{C}$-Borel transform is a linear isomorphism of $\mathscr{H}'_{\mathscr{C}}(C)$ onto the space of admissible entire functions of $\mathscr{C}$-type on C. Further, the $\mathscr{C}$-analytic functional $\sigma\in\mathscr{H}'_{\mathscr{C}}(C)$ is carried by the open disk $s(\tau)$ if and only if there exists a τ', $0<\tau'<\tau$, such that $\hat{\sigma}\in\{\mathscr{C}, \tau'\}$.*

4. **Extension to C^n.** The preceding theory may be generalized by extension to higher dimensions. To this end, let

$$\mathscr{C}^{(1)}, \mathscr{C}^{(2)}, \cdots, \mathscr{C}^{(n)} \tag{17}$$

be n entire functions of the form (1) and

$$\mathscr{C}^{(1)^*}, \mathscr{C}^{(2)^*}, \cdots, \mathscr{C}^{(n)^*} \tag{18}$$

the functions corresponding to (2). For

$$Z = (z_1, \cdots, z_n)\in C^n, \tag{19}$$

define

$$\begin{aligned}\mathscr{C}(Z) &= \mathscr{C}^{(1)}(z_1)\mathscr{C}^{(2)}(z_2)\cdots\mathscr{C}^{(n)}(z_n)\\ &= \sum_{\alpha\in N^n} c_\alpha Z^\alpha = \sum_{\alpha_k\in N} c_{\alpha_1}c_{\alpha_2}\cdots c_{\alpha_n} z_1^{\alpha_1}z_2^{\alpha_2}\cdots z_n^{\alpha_n}.\end{aligned} \tag{20}$$

DEFINITION 7. A function f on C^n is admissible if

$$f(Z) = \sum_{\alpha\in N^n} f_\alpha Z^\alpha \tag{21}$$

where

$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \tag{22}$$

with

$$\operatorname{supp}(f_\alpha) \subset \operatorname{supp}(c_\alpha) \tag{23}$$

and

$$Z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}\cdots z_n^{\alpha_n}. \tag{24}$$

DEFINITION 8. An admissible entire function f belongs to class $\{\mathscr{C}, \mathrm{T}\}$ if, for some $\mathrm{T}=(\tau_1, \tau_2, \cdots, \tau_n)$, $\tau_k>0$, $k=1, \cdots, n$, there exists a constant $M>0$ such that

$$|f(Z)| \leqq M\mathscr{C}^{(1)*}(\tau_1\,|z_1|)\mathscr{C}^{(2)*}(\tau_2\,|z_2|)\cdots\mathscr{C}^{(n)*}(\tau_n\,|z_n|). \tag{25}$$

We denote by $\{\mathscr{C}, \mathrm{T}\}^\sim$ the union of classes $\{\mathscr{C}, \mathrm{T}'\}$, $\tau_k'<\tau_k$, $k=1, \cdots, n$. Analogous to the case in the complex plane, we have that if $f\in\{\mathscr{C}, \mathrm{T}\}^\sim$, then

$$|f_\alpha/c_\alpha| \leqq M\mathrm{T}'^\alpha \tag{26}$$

for all α satisfying (23).

We let $\mathscr{H}_{\mathscr{C}}(s(\mathrm{T}))$ be the space of admissible holomorphic functions on the polydisk $s(\mathrm{T})$.

The extension of Theorem 1 to C^n can then be stated as follows.

THEOREM 3. *Let* $\mathrm{T}=(\tau_1, \cdots, \tau_n)$, $\tau_k>0$, $k=1, \cdots, n$. *For any* $f\in\{\mathscr{C}, \mathrm{T}\}^\sim$, *the linear functional on the space* $\mathscr{P}(C^n)$ *of polynomials in n complex variables defined by*

$$\langle f, P\rangle = \sum_{\alpha\in\operatorname{supp}(p_\alpha)} \frac{f_\alpha}{c_\alpha} p_\alpha, \tag{27}$$

where $P(z)=\sum_{\alpha\in\operatorname{supp}(p_\alpha)} p_\alpha Z^\alpha \in \mathscr{P}(C^n)$, *can be extended uniquely to a continuous linear functional* σ_f *on* $\mathscr{H}_{\mathscr{C}}(s(\mathrm{T}))$. *Further, the mapping*

$$f \to \sigma_f \tag{28}$$

is a vector space isomorphism of $\{\mathscr{C}, \mathrm{T}\}^{\sim}$ *onto* $\mathscr{H}'_{\mathscr{C}}(s(\mathrm{T}))$. *The inverse mapping is given by the equation*

$$f(T) = \langle \sigma_f, \mathscr{C}(W) \rangle, \tag{29}$$

$T=(t_1, \cdots, t_n)$, $W=(t_1z_1, t_2z_2, \cdots, t_nz_n)$, *where* σ_f *operates on functions of* $Z \in s(\mathrm{T})$.

5. **Applications.** When each of the functions of (17) is the exponential function in the complex plane, we have that

$$\mathscr{C}(Z) = e^{z_1}e^{z_2} \cdots e^{z_n} = \sum_{\alpha \in N^n} \frac{1}{\alpha!} Z^{\alpha} \tag{30}$$

and it is clear, with $c_\alpha = 1/\alpha!$, that all entire functions are admissible. In this case the continuous linear functionals on the space of holomorphic functions on a polydisk may be interpreted as entire functions of exponential type. Further, since $\mathscr{C}(W)=e^{\langle Z,T\rangle}$,

$$\langle Z, T\rangle = z_1t_1 + z_2t_2 + \cdots + z_nt_n,$$

the $\mathscr{C}$-Borel transform (29) can be written in the form

$$\langle \sigma_f, e^{\langle Z,T\rangle}\rangle$$

which is the classical Fourier-Borel transform.

If each of the functions of (17) is equal to

$$\mathscr{J}(z) = 2^{\nu-1/2}\Gamma(\nu + \tfrac{1}{2})z^{1/2-\nu}J_{\nu-1/2}(z),$$

J_α being the Bessel function of order α, then if

$$b_{2k} = \frac{(-1)^k\Gamma(\nu + \frac{1}{2})}{2^{2k}k!\,\Gamma(\nu + \frac{1}{2} + k)},$$

we have that

$$\mathscr{C}(Z) = \mathscr{J}(z_1)\mathscr{J}(z_2)\cdots\mathscr{J}(z_n) = \sum_{\alpha_k \text{ even}} b_{\alpha_1}b_{\alpha_2}\cdots b_{\alpha_n}z_1^{\alpha_1}\cdots z_n^{\alpha_n},$$

and it follows that all even entire functions are admissible. Here the linear functionals on the space of even holomorphic functions on a polydisk may be identified with the even entire functions of certain growth. The transform (29) in this case becomes the Hankel-Borel transform $\langle \sigma_f, \prod_{k=1}^n \mathscr{J}(z_kt_k)\rangle$.

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