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TENSOR PRODUCT MAPPINGS. II

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ABSTRACT. In this paper factorization techniques are introduced into the study of tensor product mappings to complete and improve on some results obtained by the author in an earlier paper [*Tensor product mappings*, Math. Ann. **188** (1970), 1–12. MR **44** #2052]. The main results are as follows: Let α be any $\otimes$-norm. Then

(i) if S is absolutely summing and T is an integral operator then $S \otimes_\alpha T$ is absolutely summing,

(ii) if S is quasi-nuclear and T is nuclear then $S \otimes_\alpha T$ is quasi-nuclear,

(iii) if S and T are integral operators then $S \otimes_\alpha T$ is integral.

That the results (i) and (ii) are essentially the best possible was shown by examples in the earlier quoted paper. Also, the methods developed in this paper yield a much simpler proof of the main result of the earlier paper.

The purpose of this paper is to continue the investigation of tensor product mappings begun in [**4**] and to demonstrate the use of factorization techniques in this study. In [**4**] various permanence properties of certain classes of operators were proved and contrasts in the results obtained between the two crossnorms ε and π on the tensor product were discussed. In particular it was shown that if S and T are absolutely summing operators (resp., quasi-nuclear operators) then $S \otimes_\varepsilon T$ is absolutely summing (resp., quasi-nuclear). However, simple examples show that for each of these cases $S \otimes_\pi T$ need not have the property. In this paper we show that by simply requiring one of S or T to satisfy a slightly stronger condition we can achieve permanence of the above properties for *any* $\otimes$-norm [**2**] on the tensor product. More precisely,

(i) *if S is absolutely summing and T is an integral operator* [**1**] *then $S \otimes_\alpha T$ is absolutely summing for every $\otimes$-norm α*;

(ii) *if S is a quasi-nuclear and T is nuclear then $S \otimes_\alpha T$ is quasi-nuclear for every $\otimes$-norm α.*

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An examination of the proof of (i) also establishes

(iii) *if S and T are integral operators then $S \otimes_\alpha T$ is integral for every $\otimes$-norm α.*

In addition to these results we give a greatly simplified proof of the main theorem of [**4**].

Our results depend upon the introduction of factorization techniques into the study of tensor product mappings and upon known factorization theorems for the classes of operators we study. The simplicity of this viewpoint in the investigation of certain problems connected with tensor product mappings will, it is hoped, be amply demonstrated.

Throughout the paper E_1, E_2, F_1 and F_2 will denote Banach spaces and the term "operator" or "map" will always refer to a continuous linear transformation. We often indicate that A is an operator from E to F by $E \to^A F$. Any unexplained notation or concept will be as in [**4**]. We also refer the reader to this paper for the necessary background information concerning the classes of operators under discussion.

Since it does not appear in [**4**] but is crucial to our work we give here the definition of an integral operator (see [**1**] for a complete discussion of this notion).

Definition. An operator $T: E \to F$ is said to be *integral* if T has the factorization

$$T: E \longrightarrow C(K) \xrightarrow{i} L^1(\mu) \longrightarrow F^{**},$$

where K is the unit ball in E^*, μ is a Borel measure on K, and i is the injection map.

It is well known that every integral map is absolutely summing [**1**], [**9**], and that an absolutely summing map whose range is in $\ell^\infty(\Gamma)$ for some set Γ is integral [**10**].

Theorem 1. *Let $S: E_1 \to E_2$ be an absolutely summing operator and $T: F_1 \to F_2$ an integral operator. Then $S \otimes_\alpha T: E_1 \otimes_\alpha F_1 \to E_2 \otimes_\alpha F_2$ is absolutely summing for any $\otimes$-norm α.*

Proof. Let $j: E_2 \to \ell^\infty(\Gamma)$ be the natural isometry of E_2 into the space of bounded functions on the unit ball Γ in E_2^*. Then from the remarks above $j \circ S$ is integral and hence has the factorization

$$j \circ S: E_1 \xrightarrow{A_1} C(K_1) \xrightarrow{i_1} L^1(\mu) \xrightarrow{B_1} \ell^\infty(\Gamma).$$

Similarly, since T is integral it can be written

$$T: F_1 \xrightarrow{A_2} C(K_2) \xrightarrow{i_2} L^1(\nu) \xrightarrow{B_2} F_2^{**}.$$

Let $Q=i_2 \circ A_2$ and consider the mapping $(j \circ S)\otimes Q$. Clearly this map factors as

$$(j \circ S) \otimes Q : E_1 \otimes_\alpha F_1 \xrightarrow{A_1 \otimes_\alpha A_2} C(K_1) \otimes_\alpha C(K_2)$$
$$\xrightarrow{i_1 \otimes i_2} L^1(\mu) \otimes_\alpha L^1(\nu) \xrightarrow{B_1 \otimes I} \ell^\infty(\Gamma) \otimes_\alpha L^1(\nu),$$

where I denotes the identity operator.

Recall that $\varepsilon \leqq \alpha \leqq \pi$ and that $C(K_1) \otimes_\varepsilon C(K_2)=C(K_1 \times K_2)$, while $L^1(\mu) \otimes_\pi L^1(\nu)=L^1(\mu \times \nu)$ [1]. Thus the operator $(B_1 \otimes I) \circ (i_1 \otimes i_2)$ factors as

$$(B_1 \otimes I) \circ (i_1 \otimes i_2): C(K_1) \otimes_\alpha C(K_2) \longrightarrow C(K_1) \otimes_\varepsilon C(K_2)$$
$$\xrightarrow{i} L^1(\mu) \otimes_\pi L^1(\nu) \xrightarrow{B_1 \otimes I} \ell^\infty(\Gamma) \otimes_\pi L^1(\nu),$$

where i is the injection of $C(K_1 \times K_2)$ into $L^1(\mu \times \nu)$ and is integral. This implies that $(j \circ S)\otimes Q: E_1 \otimes_\alpha F_1 \rightarrow \ell^\infty(\Gamma) \otimes_\pi L^1(\nu)$ is integral, hence absolutely summing. Since clearly it also factors as

$$(j \circ S) \otimes Q: E_1 \otimes_\alpha F_1 \longrightarrow E_2 \otimes_\pi L^1(\nu) \xrightarrow{j \otimes_\pi I} \ell^\infty(\Gamma) \otimes_\pi L^1(\nu),$$

where $j \otimes_\pi I$ is an isometry [3], it follows that $S \otimes Q: E_1 \otimes_\alpha F_1 \rightarrow E_2 \otimes_\pi L^1(\nu)$ is also absolutely summing, as is $S \otimes T=(I \otimes B_2) \circ (S \otimes Q)$: $E_1 \otimes_\alpha F_1 \rightarrow E_2 \otimes_\pi F_2^{**}$. Since $E_2 \otimes_\pi F_2$ is isometrically embedded in $E_2 \otimes_\pi F_2^{**}$ [1] we must have that $S \otimes T: E_1 \otimes_\alpha F_1 \rightarrow E_2 \otimes_\pi F_2$ is an absolutely summing operator. But then certainly $S \otimes_\alpha T: E_1 \otimes_\alpha F_1 \rightarrow E_2 \otimes_\alpha F_2$ is also absolutely summing (since $\alpha \leqq \pi$).

Simple modifications of the proof of Theorem 1 also establish the following two results.

THEOREM 2. *If $S: E_1 \rightarrow E_2$ and $T: F_1 \rightarrow F_2$ are both integral operators then $S \otimes_\alpha T: E_1 \otimes_\alpha F_1 \rightarrow E_2 \otimes_\alpha F_2$ is an integral operator for any $\otimes$-norm α.*

THEOREM 3. *If $S: E_1 \rightarrow L^1(\mu)$ and $T: F_1 \rightarrow F_2$ are both absolutely summing then $S \otimes_\alpha T: E_1 \otimes_\alpha F_1 \rightarrow L^1(\mu) \otimes_\alpha F_2$ is absolutely summing for any $\otimes$-norm α.*

We now apply a similar technique to show that the tensor product of a quasi-nuclear and a nuclear map is quasi-nuclear.

THEOREM 4. *Let $S: E_1 \rightarrow E_2$ be a quasi-nuclear operator and $T: F_1 \rightarrow F_2$ a nuclear operator. Then $S \otimes_\alpha T: E_1 \otimes_\alpha F_1 \rightarrow E_2 \otimes_\alpha F_2$ is quasi-nuclear for any $\otimes$-norm α.*

PROOF. As in the proof of Theorem 1, let $j: E_2 \rightarrow \ell^\infty(\Gamma)$ be the canonical isometry. Since S is quasi-nuclear the mapping $j \circ S: E_1 \rightarrow \ell^\infty(\Gamma)$ is nuclear

[8] and hence can be factored as

$$j \circ S : E_1 \xrightarrow{A_1} c_0 \xrightarrow{q_1} \ell^1 \xrightarrow{B_1} \ell^\infty(\Gamma),$$

where q_1 is the nuclear operator defined by $q_1(c_i) = (a_i c_i)$ for some $(a_i) \in \ell^1$ [10].

Similarly T, being nuclear, can be factored as

$$T : E_2 \xrightarrow{A_2} c_0 \xrightarrow{q_2} \ell^1 \xrightarrow{B_2} F_2,$$

where q_2 is defined by $q_2(c_i) = (b_i c_i)$ for $(b_i) \in \ell^1$.

Let $R = q_2 \circ A_2 : E_2 \to \ell^1$. Then $(j \circ S) \otimes R$ has the factorization

$$(j \circ S) \otimes R : E_1 \otimes_\alpha F_1 \xrightarrow{A_1 \otimes A_2} c_0 \otimes_\alpha c_0 \xrightarrow{q_1 \otimes q_2} \ell^1 \otimes_\alpha \ell^1 \xrightarrow{B_1 \otimes I} \ell^\infty(\Gamma) \otimes_\alpha \ell^1.$$

Now $q_1 \otimes q_2$ is the mapping defined by $q_1 \otimes q_2(e_i \otimes e_j) = a_i b_j e_i \otimes e_j$ (where (e_i) denotes the unit vector basis for c_0 and for ℓ^1). One checks easily that $q_1 \otimes q_2$ actually takes $c_0 \otimes_\varepsilon c_0$ into $\ell^1 \otimes_\pi \ell^1$ and is nuclear. Hence we have the factorization

$$q_1 \otimes q_2 : c_0 \otimes_\alpha c_0 \to c_0 \otimes_\varepsilon c_0 \to \ell^1 \otimes_\pi \ell^1 \to \ell^1 \otimes_\alpha \ell^1,$$

implying that $(j \circ S) \otimes R$ is nuclear, having the factorization

$$(j \circ S) \otimes R : E_1 \otimes_\alpha F_1 \longrightarrow c_0 \otimes_\alpha c_0 \longrightarrow c_0 \otimes_\varepsilon c_0 \xrightarrow{Q} \ell^1 \otimes_\pi \ell^1 \xrightarrow{B_1 \otimes I} \ell^\infty(\Gamma) \otimes_\pi \ell^1,$$

where Q is nuclear. As in the proof of Theorem 1, since j is an isometry we conclude that $E_2 \otimes_\pi \ell^1$ is isometric to a subspace of $\ell^\infty(\Gamma) \otimes_\pi \ell^1$ [3] and hence the mapping $S \otimes R : E_1 \otimes_\alpha F_1 \to E_2 \otimes_\pi \ell^1$ is quasi-nuclear.

But then $S \otimes T = (I \otimes B_2) \circ (S \otimes R) : E_1 \otimes_\alpha F_1 \to E_2 \otimes_\pi F_2$ is quasi-nuclear, implying $S \otimes_\alpha T : E_1 \otimes_\alpha F_1 \to E_2 \otimes_\alpha F_2$ is also quasi-nuclear.

Again, a slight modification of the proof of Theorem 4 shows

THEOREM 5. *Let $S : E_1 \to L^1(\mu)$ and $T : F_1 \to F_2$ be quasi-nuclear operators. Then $S \otimes_\alpha T$ is quasi-nuclear for every $\otimes$-norm α.*

Finally, using the techniques demonstrated above we give a much simpler proof of the following result which was proved in [4]:

If $S : E_1 \to E_2$ and $T : F_1 \to F_2$ are p-absolutely summing operators then $S \otimes_\varepsilon T : E_1 \otimes_\varepsilon F_1 \to E_2 \otimes_\varepsilon F_2$ is p-absolutely summing.

PROOF. Let Γ_1 and Γ_2 denote the unit balls in E_2^* and F_2^*, respectively. Then, as above, let $j_1 : E_2 \to \ell^\infty(\Gamma_1)$ and $j_2 : F_2 \to \ell^\infty(\Gamma_2)$ be the canonical isometries. Since S and T are p-absolutely summing it follows easily from [6] and [5] that we can write

$$j \circ S: E_1 \longrightarrow C(K_1) \xrightarrow{i_1} L^p(\mu) \longrightarrow \ell^\infty(\Gamma_1),$$

$$j_2 \circ T: F_1 \longrightarrow C(K_2) \xrightarrow{i_2} L^p(\nu) \longrightarrow \ell^\infty(\Gamma_2),$$

where K_1 and K_2 are the unit balls in E_1^* and F_1^* and i_1 and i_2 are injection maps.

It follows that

$$(j_1 \circ S) \otimes_\varepsilon (j_2 \circ T): E_1 \otimes_\varepsilon E_2 \longrightarrow C(K_1) \otimes_\varepsilon C(K_2)$$
$$\xrightarrow{i_1 \otimes i_2} L^p(\mu) \otimes_\varepsilon L^p(\nu) \longrightarrow \ell^\infty(\Gamma_1) \otimes_\varepsilon \ell^\infty(\Gamma_2).$$

Let α_p denote the crossnorm on $L^p(\mu) \otimes L^p(\nu)$ for which

$$L^p(\mu) \otimes_{\alpha_p} L^p(\nu) = L^p(\mu \times \nu).$$

Then $i_1 \otimes i_2$ factors as

$$i_1 \otimes i_2: C(K_1) \otimes_\varepsilon C(K_2) \xrightarrow{j} L^p(\mu) \otimes_{\alpha_p} L^p(\nu) \longrightarrow L^p(\mu) \otimes_\varepsilon L^p(\nu),$$

recalling that $C(K_1) \otimes_\varepsilon C(K_2) = C(K_1 \times K_2)$ and j is the injection map of $C(K_1 \times K_2)$ into $L^p(\mu \times \nu)$, a p-absolutely summing operator [7]. It follows that $(j_1 \circ S) \otimes_\varepsilon (j_2 \circ T)$ is p-absolutely summing. But since $E_2 \otimes_\varepsilon F_2$ is isometrically embedded in $\ell^\infty(\Gamma_1) \otimes_\varepsilon \ell^\infty(\Gamma_2)$ it then follows that $S \otimes_\varepsilon T$ is also p-absolutely summing.

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