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GENERALIZED CONTRACTIONS AND FIXED POINT THEOREMS

CHI SONG WONG[1]

ABSTRACT. Let T be a self-mapping on a complete metric space (X, d). Then T has a fixed point if there exist self-mappings $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ on $[0, \infty)$ such that (a) $\alpha_1(t)+\alpha_2(t)+\alpha_3(t)+\alpha_4(t)+\alpha_5(t)<t$ for $t>0$, (b) each α_1 is upper semicontinuous from the right, (c)

$$d(T(x), T(y)) \leqq a_1 d(x, T(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) + a_4 d(y, T(x)) + a_5 d(x, y)$$

for all pairs of distinct x, y in X, where $a_i=\alpha_i(d(x, y))/d(x, y)$. Related results are obtained for two mappings and mappings on a bounded convex subset of a uniformly convex Banach space.

0. **Introduction.** Let (X, d) be a (nonempty) complete metric space. Let T be a self-mapping on X. It is the purpose of this paper to obtain some fixed point theorems for certain T which are controlled by five functions α_i's from $(0, \infty)$ to $[0, \infty)$: for any distinct x, y in X,

$$d(T(x), T(y)) \leqq a_1 d(x, T(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) + a_4 d(y, T(x)) + a_5 d(x, y),$$

where $a_i=\alpha_i(d(x, y))/d(x, y)$. These control functions will satisfy certain regular and contractive conditions such as the upper semicontinuity from the right for each α_i and

$$\sum_{i=1}^{5} \alpha_i(t) < t \quad \text{for } t > 0.$$

Most of the results can be modified for two self-mappings S, T on X which are controlled by five functions α_i's in the above fashion (with

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$T(x)$ replaced by $S(x)$). Among others, our results generalize certain results of D. W. Boyd and J. S. W. Wong [1], Felix E. Browder ([2], [3]), G. Göhde [5]. When T is continuous and each α_i is increasing, we can characterize T topologically.

1. Self-mappings on a complete metric space.

THEOREM 1. *Let T be a self-mapping on a complete metric space (X, d). Suppose that there exist functions α_i, $i=1, 2, 3, 4, 5$, of $(0, \infty)$ into $[0, \infty)$ such that*

(a) *each α_i is upper semicontinuous from the right*;

(b) $\alpha_1(t)+\alpha_2(t)+\alpha_3(t)+\alpha_4(t)+\alpha_5(t)<t$, $t>0$;

(c) *for any distinct x, y in X,*

$$d(T(x), T(y)) \leqq a_1 d(x, T(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) + a_4 d(y, T(x)) + a_5 d(x, y),$$

where $a_i=\alpha_i(d(x, y))/d(x, y)$.

Then T has a unique fixed point.

PROOF. Let x_0 be a point in X. Define

$$x_{n+1} = T(x_n), \qquad b_n = d(x_n, x_{n+1}), \qquad n = 0, 1, 2, \cdots.$$

We shall first prove that T has a fixed point. So we may assume that $b_n>0$ for each n. By (c),

$$\begin{aligned} b_0 b_1 &= b_0 d(T(x_1), T(x_0)) \\ &\leqq \alpha_1(b_0)b_1 + \alpha_2(b_0)b_0 + \alpha_4(b_0)d(x_0, x_2) + \alpha_5(b_0)b_0. \end{aligned} \tag{1}$$

Since $d(x_0, x_2)\leqq b_0+b_1$, we have from (1)

$$b_1 \leqq \frac{\alpha_2(b_0) + \alpha_4(b_0) + \alpha_5(b_0)}{b_0 - \alpha_1(b_0) - \alpha_4(b_0)}\, b_0. \tag{2}$$

Similarly,

$$b_2 \leqq \frac{\alpha_1(b_1) + \alpha_3(b_1) + \alpha_5(b_1)}{b_1 - \alpha_2(b_1) - \alpha_3(b_1)}\, b_1. \tag{3}$$

By symmetry of x, y in (c), we may assume that $\alpha_1=\alpha_2$ and $\alpha_3=\alpha_4$. So from (2), (3) and induction,

$$b_{n+1} \leqq \alpha(b_n), \qquad n = 0, 1, 2, \cdots, \tag{4}$$

where

$$\alpha(t) = \frac{\alpha_1(t) + \alpha_3(t) + \alpha_5(t)}{t - \alpha_2(t) - \alpha_4(t)}\, t, \qquad t > 0.$$

From (b), $\alpha(t)<t$ for $t>0$. So $\{b_n\}$ is decreasing and therefore converges to some point b in $[0, \infty)$. If $b>0$, then

$$b = \lim_{n\to\infty} b_{n+1} \leqq \limsup_{n\to\infty} \alpha(b_n). \tag{5}$$

From (a), α is upper semicontinuous from the right. So from (5), $b\leqq\alpha(b)$, a contradiction. So $b=0$. We shall prove that $\{x_n\}$ is Cauchy. Suppose not. Then there exist $r>0$ and sequences $\{p(n)\}$, $\{q(n)\}$ such that for each $n=0, 1, 2, \cdots$,

$$p(n) > q(n) > n, \qquad d(p(n), q(n)) \geqq r, \tag{6}$$

and (by the well-ordering principle)

$$d(x_{p(n)-1}, x_{q(n)}) < r. \tag{7}$$

Let $n\geqq 0$, $c_n=d(x_{p(n)}, x_{q(n)})$. Then

$$r \leqq c_n \leqq d(x_{p(n)-1}, x_{q(n)}) + d(x_{p(n)-1}, x_{p(n)}) \leqq r + b_{p(n)-1}.$$

Since $\{b_n\}$ converges to 0, $\{c_n\}$ converges to r from the right. By (c),

$$\begin{aligned} c_n d(T(x_{p(n)}), T(x_{q(n)})) \leqq\ & \alpha_1(c_n)b_{p(n)} + \alpha_2(c_n)b_{q(n)} \\ & + \alpha_3(c_n)d(x_{p(n)}, x_{q(n)+1}) \\ & + \alpha_4(c_n)d(x_{q(n)}, x_{p(n)+1}) + \alpha_5(c_n)c_n. \end{aligned}$$

So by letting $n\to\infty$, we obtain

$$r^2 \leqq (\alpha_3(r) + \alpha_4(r) + \alpha_5(r))r,$$

a contradiction to (b). Hence $\{x_n\}$ is Cauchy. By completeness of (X, d), $\{x_n\}$ converges to some point x in X. We shall prove that x is a fixed point of T. Since each $b_n>0$, there is a subsequence $\{x_{k(n)}\}$ of $\{x_n\}$ such that $x_{k(n)}\neq x$ for each n. Let $n\geqq 0$, $d_n=d(x, x_{k(n)})$. Then from (c),

$$\begin{aligned} d(x_{k(n)+1}, T(x)) &= d(T(x_{k(n)}), T(x)) \\ &\leqq [\alpha_1(d_n)b_{k(n)} + \alpha_2(d_n)d(x, T(x)) + \alpha_3(d_n)d(x_{k(n)}, T(x)) \\ &\qquad + \alpha_4(d_n)d(x, x_{k(n)+1}) + \alpha_5(d_n)d_n]/d_n. \end{aligned}$$

So

$$d(x, T(x)) \leqq \frac{\alpha_2(d_n) + \alpha_3(d_n)}{d_n} d(x, T(x)) + o(n),$$

where $\{o(n)\}$ converges to 0. Since $\alpha_2(t)+\alpha_3(t)<t/2$ for $t>0$,

$$d(x, T(x)) \leqq d(x, T(x))/2.$$

So $T(x)=x$. If T has two distinct fixed points x, y in X, then

$$d(x,y) = d(T(x), T(y)) \leqq (\alpha_3(d(x,y)) + \alpha_4(d(x,y)) + \alpha_5(d(x,y))) < d(x,y),$$

a contradiction. Hence T has a unique fixed point in X.

When $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0$, Theorem 1 is reduced to a result of D. W. Boyd and J. S. W. Wong [**1**, Theorem 1]. We expand their proof by considering the functions $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha$ instead of a single mapping α.

THEOREM 2. *Let S, T be self-mappings on a complete metric space (X, d). Suppose that there exist upper semicontinuous functions $\alpha_1=\alpha_2$, $\alpha_3=\alpha_4$, α_5 of* $(0, \infty)$ *into* $[0, \infty)$ *such that*

(a) $\alpha_1(t)+\alpha_2(t)+\alpha_3(t)+\alpha_4(t)+\alpha_5(t)<t$, $t>0$;

(b) *for any distinct x, y in X,*

$$d(S(x), T(y)) \leqq a_1 d(x, S(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) + a_4 d(y, S(x)) + a_5 d(x,y),$$

where $a_i=\alpha_i(d(x,y))/d(x,y)$.

Then either S or T has a fixed point.

OUTLINE OF A PROOF. Let x_0 be a point in X. Define

$$x_{2n+1} = T(x_{2n}), x_{2n+2} = S(x_{2n+1}), \qquad n = 0, 1, 2, \cdots.$$

We may assume that $d(x_n, x_{n+1})>0$ for each n, otherwise some x_n is a fixed point of S or T. With the notation of Theorem 1, (1)–(7) still hold. However we cannot write $d(x_{p(n)+1}, x_{q(n)+1})$ as $d(S(x_{p(n)}), T(x_{q(n)}))$ or $d(S(x_{q(n)}), T(x_{p(n)}))$ unless $p(n)+q(n)$ is odd. When $p(n)+q(n)$ is even, we may estimate $d(x_{p(n)+1}, x_{q(n)+1})$ by

$$d(S(x_{q(n)}), T(x_{p(n)-1})) + d(x_{p(n)}, x_{p(n)+1})$$

or

$$d(S(x_{p(n)-1}), T(x_{q(n)})) + d(x_{p(n)}, x_{p(n)+1}).$$

But then $\{d(x_{p(n)-1}, x_{q(n)})\}$ converges to r from the left. So we assume that each α_i is upper semicontinuous.

There are simple examples of S, T which satisfy the conditions of Theorem 2, but S has two fixed points and T has none.

2. **Approximations and errors.**

THEOREM 3. *In Theorem* 1, *suppose further that each α_i is increasing. Then*

(i) $d(T^n(x), x_0)\leqq\alpha^n(d(x, x_0))$, $x \in X$, $n=0, 1, 2, \cdots$, *where x_0 is the* *fixed point of* T, $\alpha(0)=0$ *and for* $t>0$,

$$\alpha(t) = \frac{\alpha_1(t) + \alpha_2(t) + \alpha_3(t) + \alpha_4(t) + 2\alpha_5(t)}{2t - \alpha_1(t) - \alpha_2(t) - \alpha_3(t) - \alpha_4(t)} t;$$

(ii) α *is increasing, continuous from the right and for any* t *in* $[0, \infty)$, $\{\alpha^n(t)\}$ *converges to* 0. *Hence* $\{T^n(x)\}$ *converges uniformly to the fixed point of* T *on any bounded subset of* X.

PROOF. (i) Let x be a point in $X\backslash\{x_0\}$, $n\geqq 0$, $b_n=d(T^n(x), x_0)$. By (c) in Theorem 1,

$$\begin{aligned} b_0 b_1 &\leqq b_0 d(T(x), T(x_0)) \\ &\leqq \alpha_1(b_0)d(x, T(x)) + \alpha_3(b_0)b_0 + \alpha_4(b_0)b_1 + \alpha_5(b_0)b_0 \\ &\leqq \alpha_1(b_0)(b_0 + b_1) + \alpha_3(b_0)b_0 + \alpha_4(b_0)b_1 + \alpha_5(b_0)b_0. \end{aligned}$$

So

$$b_1 \leqq \frac{\alpha_1(b_0) + \alpha_3(b_0) + \alpha_5(b_0)}{b_0 - \alpha_1(b_0) - \alpha_4(b_0)} b_0. \tag{8}$$

Similarly (interchange the roles of x, x_0),

$$b_2 \leqq \frac{\alpha_2(b_1) + \alpha_4(b_1) + \alpha_5(b_1)}{b_1 - \alpha_2(b_1) - \alpha_3(b_1)} b_1. \tag{9}$$

Because of the symmetry of x, y, (c) in Theorem 1 still holds if we replace α_1, α_2, α_3, α_4, α_5 respectively by

$$\frac{\alpha_1 + \alpha_2}{2}, \quad \frac{\alpha_1 + \alpha_2}{2}, \quad \frac{\alpha_3 + \alpha_4}{2}, \quad \frac{\alpha_3 + \alpha_4}{2}, \quad \alpha_5.$$

Thus from (8), (9) and induction, $b_{n+1}=\alpha(b_n)$, $n=0, 1, 2, \cdots$. So by induction

$$d(T^n(x), x_0) = b_n \leqq \alpha^n(b_0) = \alpha^n(d(x, x_0)), \qquad n = 0, 1, 2, \cdots.$$

(ii) Each α_i is increasing and continuous from the right, so is α. Let $t>0$. By (b) in Theorem 1, $\alpha(t)<t$. So $\{\alpha^n(t)\}$ is decreasing and therefore converges to some t_0 in $[0, \infty)$. $t_0=0$, otherwise, by the right continuity of α,

$$t_0 = \lim_{n\to\infty} \alpha^{n+1}(t) \leqq \alpha\left(\lim_{n\to\infty} \alpha^n(t)\right) = \alpha(t_0),$$

a contradiction to $\alpha(t)<t$ for $t>0$.

When $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0$ and X is bounded, Theorem 3 is reduced to a result of Felix E. Browder [3, Theorem 1] with $M=X$. As Browder pointed out, it is the explicit control over the error term in the Picard theorem which contributes so much to the widespread usefulness. Theorem 3 gives a uniform error control sequence $\{\alpha^n(d_0)\}$ over the error sequence $\{d(T^n(x), x_0)\}$ of functions $d(T^n(x), x_0)$ on any bounded subset of X with a diameter d_0.

3. **Topological contractions.** Let $(X, \mathcal{T})$ be a metrizable topological space. Suppose that $\mathcal{T}$ is metrically topologically complete, i.e. $\mathcal{T}$ admits a complete metric for X. A self-mapping T on X is a topological contraction if there is an admissible complete metric d for X such that T is a contraction with respect to d, i.e. there exists a number k in $[0, 1)$ such that $d(T(x), T(y)) \leqq kd(x, y)$ for all x, y in X. Very often, a complete metric space (X, d) is given and then a topology $\mathcal{T}$ for X is induced; is there a practical way of recognizing topological contractions on $(X, \mathcal{T})$ in the language of d? The following result gives a partial answer to this question.

THEOREM 4. *In Theorem* 3, *suppose further that T is continuous. Then T is a topological contraction on X.*

PROOF. From Theorems 1 and 3, T has a unique fixed point x_0 and for any x in X, $\{T^n(x)\}$ converges to x_0. By a result of P. R. Meyer [**6**, Theorem 1], we need only to find a neighborhood V of x_0 such that $\{T^n(V)\}$ converges to x_0. Let V be the open ball in X with center at x_0 and radius 1. Then for any x in V, $d(T^n(x), x_0) \leqq \alpha^n(1)$ for each x in V. So for any neighborhood U of x_0, $T^n(V)$ is included in U for large n's.

4. **Self-mappings on a bounded closed convex subset of a uniformly convex Banach space.** In Theorem 1, it can be proved that if X is compact and if "$<$" in (b) is interchanged with "$\leqq$" in (c), then T has a unique fixed point. However, compactness is too strong for many purposes. The following result is perhaps more interesting.

THEOREM 5. *Let K be a bounded closed convex subset of a uniformly convex Banach space B. Let S, T be continuous self-mappings on K such that either*

$$\inf\{d(S(x), x) : x \in K\} = 0 \quad or \quad \inf\{d(T(x), x) : x \in K\} = 0.$$

Suppose further that there exist functions α_i, $i=1, 2, \cdots, 5$, of $K \times K$ into $[0, \infty)$ such that $\alpha_1=\alpha_2$, $\alpha_3=\alpha_4$, $\sum_{i=1}^5 \alpha_i \leqq 1$ and for any distinct x, y in K,

$$d(S(x), T(y)) \leqq a_1 d(x, S(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) + a_4 d(y, T(x)) + a_5 d(x, y),$$

where $a_i = \alpha_i(x, y)$.

Then S or T has a fixed point.

PROOF. By symmetry, we may assume that

$$\inf\{d(S(x), x) : x \in K\} = 0.$$

So there is a sequence $\{x_n\}$ in K for which $\{d(x_n, S(x_n))\}$ converges to 0, where d is the metric for B induced by the norm $\| \ \|$ for B. Let

$$r_m(y) = \sup\{d(y, x_n) : n \geqq m\}, \qquad m = 0, 1, 2, \cdots. \tag{10}$$

Since K is uniformly convex, there exists a unique c_m in K such that

$$r_m(c_m) = \inf\{r_m(y) : y \in K\}. \tag{11}$$

By a recent result of M. Edelstein [4, Theorem 1], $\{c_m\}$ converges to some point c in K. c is called the asymptotic center of $\{x_n\}$. If $\{x_n\}$ converges to c, then by continuity of S, c is a fixed point of S. Suppose that $\{x_n\}$ does not converge to c. We shall prove that c is a fixed point of T. Let $n \geqq m \geqq 1$. Then with $a_i = \alpha_i(x_n, c_m)$, we have

$$d(S(x_n), T(c_m)) \leqq a_1 d(x_n, S(x_n)) + a_2 d(c_m, T(c_m)) + a_3 d(x_n, T(c_m)) + a_4 d(c_m, S(x_n)) + a_5 d(x_n, c_m).$$

Since

$$d(S(x_n), T(c_m)) \geqq d(x_n, T(c_m)) - d(S(x_n), x_n),$$
$$d(c_m, S(x_n)) \leqq d(c_m, x_n) + d(x_n, S(x_n)),$$
$$d(c_m, T(c_m)) \leqq d(x_n, c_m) + d(x_n, T(c_m)),$$

we have

$$d(x_n, T(c_m)) \leqq (a_2 + a_4 + a_5) d(x_n, c_m) + (a_2 + a_3) d(x_n, T(c_m)) + (1 + a_1 + a_4) d(x_n, S(x_n)).$$

So

$$\begin{aligned} d(x_n, T(c_m)) &\leqq \frac{a_2 + a_4 + a_5}{1 - a_2 - a_3} d(x_n, c_m) + \frac{1 + a_1 + a_4}{1 - a_2 - a_3} d(x_n, S(x_n)) \\ &\leqq d(x_n, c_m) + 2d(x_n, S(x_n)) \leqq r_m(c_m) + s_m, \end{aligned} \tag{12}$$

where $s_m = \sup\{2d(x_n, S(x_n)) : n \geqq m\}$. Suppose to the contrary that $T(c) \neq c$. Then by continuity of T, there exists s in $(0, M]$ and a subsequence $\{c_{k(m)}\}$ of $\{c_m\}$ such that

$$d(c_{k(m)}, T(c_{k(m)}) \geqq s \quad \text{for each } m,$$

where M is the diameter of K. Since B is uniformly convex,

$$\delta(\varepsilon) = \inf\{1 - \|x + y\|/2 : \|x\| \leqq 1, \|y\| \leqq 1, \|x - y\| \geqq \varepsilon\}$$

is positive for each ε in $(0, 2]$. So by (10), (11) and (12),

$$\left\| x_n - \frac{c_{k(m)} + T(c_{k(m)})}{2} \right\| \leqq (r_{k(m)}(c_{k(m)}) + s_{k(m)})(1 - \delta(s/M))$$

for each $n \geqq k(m)$. Since $\{s_m\}$ converges to 0,

$$\sup\left\{\left\|x_n - \frac{c_{k(m)} + T(c_{k(m)})}{2}\right\| : n \geqq k(m)\right\} < r_{k(m)}(c_{k(m)})$$

for large m, a contradiction to the choice of $r_m(c_m)$'s.

We remark here that when $S=T$ and $\alpha_1=\alpha_2=0$, the condition $\inf\{d(T(x), x) : x \in K\}$ is automatically satisfied.

THEOREM 6. *Let K be a bounded closed convex subset of a uniformly convex Banach space B. Let $\mathcal{F}$ be a commuting family of continuous self-mappings T on K which satisfy the conditions of Theorem 5 with $S=T$. Then the set of all common fixed points of $\mathcal{F}$ is nonempty, closed and convex.*

PROOF. Let T be an element of $\mathcal{F}$ and let F_T be the set of all fixed points of T. By Theorem 5, F_T is nonempty. By continuity of T, F_T is closed. Let x, y be distinct elements of F_T, let $t \in (0, 1)$ and let $z=(1-t)x+ty$. Then with the notation of Theorem 5 and with $a_i=\alpha_i(z, x)$,

$$\begin{aligned} d(T(z), x) &= d(T(z), T(x)) \\ &\leqq a_1 d(z, T(z)) + (a_3 + a_5) d(z, x) + a_4 d(x, T(z)). \end{aligned}$$

Since $d(z, T(z)) \leqq d(z, x) + d(x, T(z))$, we have

$$d(T(z), x) \leqq \frac{a_1 + a_3 + a_5}{1 - a_1 - a_4} d(z, x) \leqq d(z, x) = t d(x, y).$$

Similarly, $d(T(z), y) \leqq (1-t) d(x, y)$. Since B is strictly convex, $T(z)=z$. So F_T is convex. This proves the theorem for a single mapping. Consider the family $\mathcal{A}=\{F_f : f \in \mathcal{F}\}$. Obviously the intersection of the members of $\mathcal{A}$ is closed and convex. For any f, g in $\mathcal{F}$, f, g commute and therefore $f(F_g)$ is included in F_g; thus applying this theorem to f (restricted to F_g), we conclude that F_f intersects F_g. Repeating this argument, we know that $\mathcal{A}$ has the finite intersection property. Since B is reflexive, every bounded closed convex subset of B is weakly compact. So $\mathcal{A}$ has nonempty intersection.

If we restrict $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0$ in Theorem 5, then the above theorem is reduced to the well-known result of Browder [2] and Göhde [5].

ADDED IN PROOF. For Theorem 6 and more references, one should see the paper by K. Goebel, W. A. Kirk and T. N. Shimi: *A fixed point theorem in uniformly convex spaces.* With a different approach, Theorem 5 is proved with $S=T$ and without the condition $\inf\{d(x, T(x)); x \in K\}=0$.

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Summer Research Institute, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada