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A NOTE ON A LEMMA OF ZARISKI AND HIGHER DERIVATIONS

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ABSTRACT. A sufficient condition is given for an $\mathfrak{a}$-adic complete ring R to be a power series ring over a subring.

1. **Introduction.** We prove in this note the following theorem: Let R be a ring and let $\mathfrak{a}$ be an ideal in R such that $\bigcap_{i=0}^{\infty} \mathfrak{a}^i=0$ and R is complete with respect to the $\mathfrak{a}$-adic topology. Assume that there exists a higher derivation $D=\{D_i\}_{i=0}^{\infty}$ of R such that $D_1(x)=1$ for some $x\in\mathfrak{a}$. Let $E=D_0-xD_1+\cdots+(-1)^n x^n D_n+\cdots$. If $E(x)=0$, then there exists a subring R_1 of R such that $R=R_1[[x]]$, and x is analytically independent over R_1.

This result generalizes Zariski's original lemma [**5**, Lemma 4, p. 526], and [**1**, Theorem 6, p. 412], a version of Zariski's lemma when R is of positive characteristic, and also removes the condition that R is an integral domain as we mentioned at the end of [**1**, p. 414]. [**5**, Lemma 4, p. 256] played a very important role in the study of analytic product of an affine algebraic variety V along a given subvariety W of V in A. Seidenberg's paper on differential ideals [**4**].

In the last section, we generalize a lemma of M. Miyanishi [**2**, p. 194] slightly, and give some remarks on his proofs of his lemma and Proposition 1.3 [**2**, p. 194].

2. **Preliminaries.** Throughout this note, all rings are commutative with identity. A derivation D of a ring R is an additive group homomorphism from R to R such that $D(a\cdot b)=aDb+bDa$ for all a and b in R. A higher derivation $D=\{D_i\}_{i=0}^{\infty}$ of a ring R is a sequence of additive group homomorphisms from R to R such that

(1) D_0=identity map on R, and

(2) $D_n(a\cdot b)=\sum_{i=0}^{n} D_i(a)\cdot D_{n-i}(b)$ for all $n=1,2,\cdots$, and for all a and b in R. (Note D_1 is always a derivation of R.) *Leibniz formula.*

Let R be a ring and let $\mathfrak{a}$ be an ideal in R. Then R has a topological structure with $\{\mathfrak{a}^i\}_{i=0}^{\infty}$ as the fundamental system of neighborhoods of the

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zero in R. This is the so called $\mathfrak{a}$-adic topology in R. Both addition and multiplication are continuous in $\mathfrak{a}$-adic topology. R is a Hausdorff space if and only if $\bigcap_{i=0}^{\infty} \mathfrak{a}^i=0$. A mapping $f: R\to R$ is continuous if there exists a subsequence $\{\mathfrak{a}^{\nu_n}\}_{n=0}^{\infty}$ of the sequence $\{\mathfrak{a}^n\}_{n=0}^{\infty}$ such that $f(\mathfrak{a}^{\nu_n})\subset \mathfrak{a}^n$, for each natural number n.

LEMMA 1. *Let R be a ring. Let $D=\{D_i\}_{i=0}^{\infty}$ be a higher derivation of R. Then*

(1) *For $a_1, \cdots, a_n \in R$, and for each natural number m,*

$$\begin{aligned} D_m(a_1\cdots a_n) &= \sum_{i=1}^{n} a_1\cdots \hat{a}_i \cdots a_n D_m(a_i) \\ &\quad + \sum_{i=1}^{m-1} D_i(a_1\cdots a_{n-1})\cdot D_{m-i}(a_n) \\ &\quad + \sum_{i=1}^{m-1}\sum_{j=2}^{n-1} a_{j+1}\cdots a_n D_i(a_1\cdots a_{j-1})D_{m-i}(a_j), \end{aligned}$$

where $a_1\cdots \hat{a}_i\cdots a_n=a_1\cdots a_{i-1}a_{i+1}\cdots a_n$.

(2) *For each ideal $\mathfrak{a}$ in R, and for each natural number m, $D_m(\mathfrak{a}^n)\subset \mathfrak{a}^{n-m}$ for each $\mathfrak{a}^n$, i.e. D_m is continuous with respect to the $\mathfrak{a}$-adic topology.*

(3) *For each ideal $\mathfrak{a}$ in R and for each $x\in\mathfrak{a}$, let $E_l=\sum_{i=0}^{l}(-1)^i x^i D_i$ for each natural number l. $E_l(\mathfrak{a}^n)\subset\mathfrak{a}^n$ for each $\mathfrak{a}^n$, i.e. E_l is a continuous additive group homomorphism of R with respect to the $\mathfrak{a}$-adic topology.*

PROOF. Straightforward.

LEMMA 2. *Let R be a ring. Let $\mathfrak{a}$ be an ideal of R such that $\bigcap_{i=0}^{\infty}\mathfrak{a}^i=0$. Assume R is complete with respect to the $\mathfrak{a}$-adic topology. Let $x\in\mathfrak{a}$, and let $E_l=\sum_{i=0}^{l}(-1)^i x^i D_i$. Then the sequence $\{E_l\}_{l=0}^{\infty}$ is uniformly convergent, and $E=\sum_{i=0}^{\infty}(-1)^i x^i D_i=\lim_{l\to\infty} E_l$ is a continuous endomorphism of R.*

PROOF. The sequence $\{E_l(a)\}$ is a Cauchy sequence for each $a\in R$. In fact $E_i(a)-E_j(a)\in\mathfrak{a}^n$ for $i,j>n$; i.e., for each given $\mathfrak{a}^n$, there exists a natural number $N\,(=n)$ such that $E_i(a)-E_j(a)\in\mathfrak{a}^n$ for $i,j>N$. Hence $\{E_l(a)\}$ converges in R. Since the natural number N above is independent of $a\in R$, therefore $\{E_l\}_{l=0}^{\infty}$ converges uniformly in R. Hence $E=\lim_{l\to\infty} E_l$ is continuous. In fact for each $\mathfrak{a}^n$ and for each $a\in\mathfrak{a}^n$, there exists a natural number $N=n$ which is independent of a such that $E(a)=(E(a)-E_N(a))+E_N(a)\in\mathfrak{a}^n$. Thus $E(\mathfrak{a}^n)\subset\mathfrak{a}^n$.[1]

Since $E(a+b)=\lim_{l\to\infty}E_l(a+b)=\lim_{l\to\infty}E_l(a)+\lim_{l\to\infty}E_l(b)=E(a)+E(b)$, therefore $E(a+b)=E(a)+E(b)$. Also for each natural number l,

[1] I thank the referee for his remark that $E(\mathfrak{a}^n)\subset\mathfrak{a}^n$.

$E_l(a)\cdot E_l(b)\equiv E_l(ab)\bmod(x^{l+1})$. Thus $E_l(a)E_l(b)-E_l(ab)\in\mathfrak{a}^{l+1}$. Therefore

$$\lim_{l\to\infty} E_l(a)\cdot E_l(b) = \lim_{l\to\infty} E_l(ab).$$

But $\lim_{l\to\infty} E_l(a)\cdot E_l(b)=\lim_{l\to\infty} E_l(a)\cdot\lim_{l\to\infty} E_l(b)$. So $E(ab)=E(a)E(b)$.

3. THEOREM 1. *Let R be a ring and let $\mathfrak{a}$ be a proper ideal in R such that $\bigcap_{i=0}^{\infty}\mathfrak{a}^i=0$, and R is complete with respect to the $\mathfrak{a}$-adic topology. Assume that there exists a higher derivation $D=\{D_i\}_{i=0}^{\infty}$ of R such that $D_1(x)=1$ for some $x\in\mathfrak{a}$. Let $E=D_0-xD_1+\cdots+(-1)^nx^nD_n+\cdots$. If $E(x)=0$, then there exists a subring R_1 of R such that $R=R_1[[x]]$, and x is analytically independent over R_1; i.e. if $\sum_{i=0}^{\infty}a_ix^i=0$ where $a_i\in R_1$ then $a_i=0$ for all $i=1,2,\cdots$.*

PROOF. $E(x)=0$ implies $E^{-1}(0)=Rx$. Indeed, if $a\in E^{-1}(0)$, then $E(a)=0$; i.e. $a-xD_1(a)+\cdots+(-1)^nx^nD_n(a)+\cdots=0$. Thus $a=xD_1(a)+\cdots+(-1)^nx^nD_n(a)+\cdots$ is in Rx. So $E^{-1}(0)\subset Rx$. The other inclusion is obvious. Next, we observe that $E^2=E$. In fact for each $a\in R$,

$$E(a) = a - xD_1a + \cdots$$

and

$$\begin{aligned} E^2(a) &= E(a) - E(xD_1a + \cdots) \\ &= E(a) - E(x)\cdot E(D_1a - xD_2a + \cdots) = E(a), \end{aligned}$$

so $E^2(a)=E(a)$ for all $a\in R$. Let $R_1=E(R)$ then E is an identity map on R_1. Let a be an arbitrary element in R. $E(a)=a-xD_1a+\cdots+(-1)^nx^nD_na+\cdots$ implies that $a=E(a)+a_1x$ for some $a_1\in R$. Thus $a=E(a)+xE(a_1)+a_2x^2$ for some $a_2\in R$, and so on. Therefore we have $a=E(a)+xE(a_1)+\cdots+x^nE(a_n)+\cdots$ for some $a_1,a_2,\cdots,a_n,\cdots$ in R, and $a\in R_1[[x]]$. Hence $R=R_1[[x]]$. Finally we suppose $b_0+b_1x+\cdots+b_mx^n+\cdots=0$ where $b_0,b_1,\cdots,b_n,\cdots$ are in R_1. We prove inductively $b_0=b_1=\cdots=b_n=\cdots=0$. Since E is identity on R_1, $E(b_0+b_1x+\cdots+b_nx^n+\cdots)=0$ implies $b_0=E(b_0)=0$. Assume $b_0=b_1=\cdots=b_i=0$. We have $b_{i+1}x^{i+1}+b_{i+2}x^{i+2}+\cdots=0$. By Lemma 1 and $D_1(x)=1$, we have $D_nx^n\equiv 1\bmod(x)$ and $D_nx^{n+j}\equiv 0\bmod(x)$ for all natural numbers n and j. Thus

$$\begin{aligned} 0 &= D_{i+1}(b_{i+1}x^{i+1} + b_{i+2}x^{i+2} + \cdots) \\ &= D_{i+1}(b_{i+1}x^{i+1}) + D_{i+1}(b_{i+2}x^{i+2} + \cdots) \\ &\equiv b_{i+1}D_{i+1}(x^{i+1})\bmod(x) \equiv b_{i+1}\bmod(x). \end{aligned}$$

Therefore $b_{i+1}+c_{i+1}x=0$ for some $c_{i+1}\in R$. So $0=E(b_{i+1}+c_{i+1}x)=b_{i+1}$. Therefore x is analytically independent over R_1.

If R contains the field of rational numbers as a subring, then every derivation D of R gives rise to a higher derivation of R, namely, $\{D_0, D, D^2/2!, \cdots, D^n/n!, \cdots\}$ where D^n is the nth successive derivation of D, and D_0 is the identity map in R. Let $\mathfrak{a}$ be an ideal of R such that R is a complete Hausdorff space with respect to the $\mathfrak{a}$-adic topology. Assume $Dx=1$ for some $x \in \mathfrak{a}$. Then the endomorphism $E=e^{-xD}=\sum(-1)^n x^n D^n/n!$ always maps x to zero. Let $R_1=E(R)$, then $D(R_1)=0$. Thus we have the following

COROLLARY 1. *Let R be a ring containing the field of rational numbers as a subring. Let $\mathfrak{a}$ be an ideal in R such that R is a complete Hausdorff space with respect to the $\mathfrak{a}$-adic topology. Assume there exists a derivation D of R such that $Dx=1$ for some $x \in \mathfrak{a}$. Then there exists a subring R_1 of R such that* (1) *D is zero on R_1 and* (2) *$R=R_1[[x]]$ and x is analytically independent over R_1.*

In the following, a semilocal (local) ring $\mathfrak{D}$ is a Noetherian ring with finitely many (unique) maximal ideals. Let $\mathfrak{m}$ be the intersection of the maximal ideals of $\mathfrak{D}$. It is well known that $\bigcap_{i=0}^{\infty} \mathfrak{m}^i=0$. In this case we use $\mathfrak{m}$-adic topology for R. As a corollary to Corollary 1, we have the original lemma of Zariski.

COROLLARY 2. *Let $(\mathfrak{D}, \mathfrak{m})$ be a complete semilocal ring of characteristic zero. Let D be a derivation of $\mathfrak{D}$. Assume that there exists an element x in $\mathfrak{m}$ of $\mathfrak{D}$ such that Dx is a unit in $\mathfrak{D}$. Then $\mathfrak{D}$ contains a ring $\mathfrak{D}_1$ of representatives of the* (*complete*) *semilocal ring $\mathfrak{D}/\mathfrak{D}x$ having the following properties*: (a) *D is zero on $\mathfrak{D}_1$*; (b) *x is analytically independent on $\mathfrak{D}_1$*; (c) *$\mathfrak{D}$ is the power series ring $\mathfrak{D}_1[[x]]$.*

PROOF. Replace D by $(1/Dx)D$ and apply Corollary 1.

We also get [**1**, Theorem 6, p. 412] as a corollary to the theorem,

COROLLARY 3. *Let $(\mathfrak{D}, \mathfrak{m})$ be a complete local ring. Let $x \in \mathfrak{m}$ and let $D=\{D_i\}_{i=0}^{\infty}$ be a higher derivation of $\mathfrak{D}$ such that D_1x is a unit in $\mathfrak{D}$, and $D_ix=0$ for $i>1$. Then there exists a subring $\mathfrak{D}_1$ of $\mathfrak{D}$ such that*: (a) *$\mathfrak{D}_1$ is a complete local ring*, (b) *x is analytically independent over $\mathfrak{D}_1$, and* (c) $\mathfrak{D}=\mathfrak{D}_1[[x]]$.

PROOF. Let $D_1x=\varepsilon^{-1}$, where ε is a unit in $\mathfrak{D}$. Replacing $\{D_i\}_{i=0}^{\infty}$ by $\{\varepsilon^i D_i\}_{i=0}^{\infty}$, we may assume $D_1x=1$. Since $D_ix=0$ for $i>1$, therefore $Ex=x-xD_1x=0$ where $E=D_0-xD_1+\cdots+(-1)^n x^n D_n+\cdots$. Thus the theorem is applicable.

REMARKS. (1) Theorem 1 and Corollary 1 hold under the assumption that D_1x is a unit. The proofs are easily modified.

(2) If R has $\mathfrak{a}$ as its sole maximal ideal and is a complete Hausdorff space with respect to the $\mathfrak{a}$-adic topology, then $E(x)=0$ and $x\neq 0$ implies that $D_1(x)$ is a unit.

(3) If R is a complete Hausdorff integral domain with respect to the $\mathfrak{a}$-adic topology, then $E(x)=0$ and $x\neq 0$ implies that $D_1(x)$ is a unit.

4. Though the following theorem could be easily proved by a similar technique used in the proof of Theorem 1, we would like to prove it as a corollary to Corollary 1.

THEOREM 2. *Let R be a ring containing the field of rational numbers as a subring. Let $\mathfrak{a}$ be an ideal in R such that $\bigcap_{i=0}^{\infty} \mathfrak{a}^i=0$. Assume that there exists a derivation D of R such that* (1) *for each $y \in R$, $D^i(y)=0$ for sufficiently large i, and* (2) *$D(x)=1$ for some $x \in \mathfrak{a}$. Then* (a) *there exists a subring R_1 of R such that $R=R_1[x]$ and x is algebraically independent over R_1;* (b) *D is trivial on R_1.*

PROOF. Let $\hat{R}$ be the completion of R with respect to the $\mathfrak{a}$-adic topology. Then $\hat{R}$ is a complete Hausdorff space with respect to the topology defined by the filtration $\hat{\mathfrak{a}}\supset(\mathfrak{a}^2)^\wedge\supset\cdots\supset(\mathfrak{a}^n)^\wedge\supset\cdots$, where $(\mathfrak{a}^i)^\wedge$ is the closure of $\mathfrak{a}^i$ in $\hat{R}$. Thus $\bigcap_{i=0}^{\infty}(\mathfrak{a}^i)^\wedge=0$. Note that $(\mathfrak{a}^i)^\wedge \supsetneqq \mathfrak{a}^i\hat{R}$ in general and equality holds if $\mathfrak{a}$ is finitely generated. Let $y_1, y_2, \cdots, y_n, \cdots$ be a Cauchy sequence in R; Lemma 1 implies that $\{D^j(y_i)\}_{i=0}^{\infty}$ is also a Cauchy sequence for each j. Define $\hat{D}^j(y)=\lim_{i\to\infty} D^j(y_i)$. Then it is easy to check that $\{\hat{D}_0, \hat{D}, \cdots, \hat{D}^n/n!, \cdots\}$ forms a higher derivation in $\hat{R}$. Moreover $\hat{D}^j(\mathfrak{a}^i)^\wedge\subset(\mathfrak{a}^{i-j})^\wedge$. Indeed let $\hat{y}\in(\mathfrak{a}^i)^\wedge$ and let $y_1, \cdots, y_n, \cdots$ be a sequence in $\mathfrak{a}^i$ such that $\lim_{n\to\infty} y_n=\hat{y}$. Then $\hat{D}^j(\hat{y})=\lim_{n\to\infty} D^j(y_n)$. Since $D^j(y_n)\in\mathfrak{a}^{i-j}$, therefore $\hat{D}^j(\hat{y})\in(\mathfrak{a}^{i-j})^\wedge$. Lemma 1 and Lemma 2 are easily verified. Since the kernel of the natural ring homomorphism from R to $\hat{R}$ is $\bigcap_{i=0}^{\infty}\mathfrak{a}^i=0$, R is viewed as a subring of $\hat{R}$. $\hat{D}^i$ restricted to R is D^i so $\hat{D}x=1$ and

$$\begin{aligned}\hat{E}(x) &= x-(x/1!)\hat{D}(x)+\cdots+(-1)^n(x^n/n!)\hat{D}^n(x)+\cdots\\ &= x-(x/1!)D(x)+(x^2/2!)D^2(x)+\cdots+(-1)^n(x^n/n!)D^n(x)+\cdots\\ &= 0.\end{aligned}$$

Thus it follows from Theorem 1 that $\hat{R}=\hat{R}_1[[x]]$, where $\hat{R}_1$ is a subring of $\hat{R}$ and x is analytically independent over $\hat{R}_1$. Let $R_1=R\cap\hat{R}_1$. Then $R=R_1[x]$. Indeed, let $y\in R$. Then $y=a_0+a_1x+\cdots+a_nx^n+\cdots$ where $a_i\in\hat{R}_1$ for all a_i. Since there exists a natural number N such that $D^N(y)=0$. Therefore

$$0=\hat{D}^N(y)=N!\,a_N+\tfrac{1}{2}(N+1)!\,a_{N+1}x+\cdots.$$

Hence $a_i=0$ for all $i\geqq N$, and $y=a_0+a_1x+\cdots+a_{N-1}x^{N-1}$. It follows that $R\subset\hat{R}_1[x]$. Applying D^{N-1} to y, we have $D^{N-1}(y)=(N-1)!a_{N-1}$.

Therefore $a_{N-1} \in R$. Applying D^{N-2} to $(y - a_{N-1}x^{N-1}) = a_0 + a_1x + \cdots + a_{N-2}x^{N-2}$, we get $a_{N-2} \in R$ and so on. Consequently, $a_0, \cdots, a_{N-1}$ are all in $\hat{R}_1 \cap R = R_1$. So $R = R_1[x]$, and x is of course algebraically independent over R_1. (b) follows from Corollary 1.

We would like to thank Professor M. Miyanishi for communicating to us the following result which we also observed independently.

PROPOSITION 1. *Let R be an integral domain of characteristic* 0. *Assume there is a derivation of R such that $D^i(a)=0$ for each $a \in R$ and for sufficiently large i. Then $Dx=0$ for all units x in R.*

PROOF. Let x be a unit in R, and let $y \in R$ be such that $xy=1$. Then $xDy+yDx=0$. Suppose $Dx \neq 0$. Thus $Dy \neq 0$. Let i be the natural number such that $D^ix=0$ and $D^mx \neq 0$ for $m<i$, also let j be the natural number such that $D^jy=0$ and $D^py \neq 0$ for $p<j$. By Leibniz's formula,

$$0 = D^n(xy) = \sum_{k=0}^{n} \binom{n}{k} D^k(x) D^{n-k}(y).$$

Taking $n=i+j-2$ and assuming $i \leqq j$ we get $D^{j-1}(x) \cdot D^{i-1}(y)=0$. Hence either $D^{i-1}(x)=0$ or $D^{j-1}(y)=0$, a contradiction.

Proposition 1 completes the proof of [2, Lemma 1.4, p. 194]. One could not use [2, Proposition 1.4, p. 194] to yield a proof to the last part of [2, Proposition 1.3, p. 193]. But Proposition 1 corrects that part of the proof.

[4, Theorem 5, p. 30] locates differential ideals; [2, Proposition 1.3, p. 193] becomes interesting, we give it a slight generalization.

THEOREM 3. *Let R be an integral domain of characteristic* 0 *with a unique maximal ideal* $\mathfrak{m}$ *such that* $\bigcap_{i=0}^{\infty} \mathfrak{m}^i = 0$, *i.e.* $(R, \mathfrak{m})$ *is a local domain which may not be Noetherian. If there is a derivation D of R such that $D^i(a)=0$ for each $a \in R$ and for sufficiently large i, then* $\mathfrak{m}$ *is differential, i.e.* $D(\mathfrak{m}) \subset \mathfrak{m}$, *and $\bar{D}$ induced by D on $R/\mathfrak{m}$ is trivial.*

PROOF. Suppose $D\mathfrak{m} \not\subset \mathfrak{m}$. Then there is $x \in \mathfrak{m}$ such that $Dx=u^{-1}$, where u is a unit in R. Then $uDx=1$. Replacing D by uD, we have $(uD)^i(a)=u^iD^i(a)$ by Proposition 1. Thus $(uD)^i(a)=0$ for sufficiently large i. It follows from Theorem 2 that $R=R_1[x]$, a contradiction. The last part follows from Proposition 1.

Observing the fact that in a polynomial ring $A[x]$ the units in $A[X]$ are of the form $a_0+a_1x+\cdots+a_nx^n$ such that a_0 is a unit in A and $a_2, \cdots, a_n$ are nilpotent in A, we give two examples countering Proposition 1 when R is not an integral domain.

EXAMPLE 1. Let $R=Z/(4)[X]$, where Z is the domain of integers and X is an indeterminate over $Z/(4)$. Let D be a derivation of R such that $DX=1+2X+2X^2$. $(DX)^2=1$, $D^2X=2\neq 0$, $D^i(a)=0$ for each $a\in R$ and for large integers i.

EXAMPLE 2. Let $R=(Q[t])[X]$, where Q is the field of rational numbers, $t^2=0$ and X is an indeterminate over $Q[t]$. Let D be a derivation of R such that $DX=1+tX$. Then DX is a unit $((DX)\cdot(1-tX)=1)$ $D^2X=t\neq 0$, and $Dt=0$, $D^i(a)=0$ for each $a\in R$ and for large i.

In the setting of Theorem 2, when R is an integral domain, if there is a derivation D such that DX is a unit u for some $x\in\mathfrak{a}$, then there is a $y\in\mathfrak{a}$ $(y=x/u)$ such that $Dy=1$. What can one say in a more general case? Both Example 1 and Example 2 give negative answers. Using an idea of Professor M. Rosenlicht [**3**, Theorem 1, p. 721] we prove the following theorem.

THEOREM 4. *Let R be a ring, which contains the field of rational numbers, with an ideal $\mathfrak{a}$ such that $\bigcap_{i=0}^{\infty}\mathfrak{a}^i=0$ and R is complete with respect to the $\mathfrak{a}$-adic topology. Assume there is a derivation D of R such that Dx is a unit for some $x\in\mathfrak{a}$. Then there exists an element $y\in\mathfrak{a}$ such that $Dy=1$.*

PROOF. If Dx is a unit then $D(x/Dx)-1\in Rx\subset\mathfrak{a}$. If we can construct a Cauchy sequence $\{x_0=x/Dx, x_1,\cdots,x_n,\cdots\}$ such that $x_i\in xR$ and $Dx_i-1\in Rx^{i+1}$, then putting $y=\lim_{i\to\infty}x_i$, and since $\mathfrak{a}$ is also closed, we have $y\in\mathfrak{a}$ and $Dy=\lim_{i\to\infty}Dx_i=1$. The proposed construction goes inductively as follows: Since $D(Rx^{i+1})\subset Rx^i$, D induces a surjective R-homomorphism $\bar{D}^{(i)}:Rx^{i+1}\to Rx^i/Rx^{i+1}$ such that

$$\bar{D}^{(i)}(x^{i+1})=(i+1)x^i+Rx^{i+1}.$$

Therefore there exists $z_i\in Rx^{i+1}$ such that

$$Dz_i\equiv Dx_{i-1}-1 \bmod(Rx^{i+1})$$

for $i=1,2,\cdots$. Thus $D(x_{i-1}-z_i)-1\in Rx^{i+1}$. Putting $x_i=x_{i-1}-z_i$, we have a sequence $\{x_0=x/Dx, x_1, x_2,\cdots\}$ such that $Dx_i-1\in Rx^{i+1}\subset\mathfrak{a}^{i+1}$ for $i=0,1,2,\cdots$. For a given $\mathfrak{a}^n$, there exists a positive integer N $(=n)$ such that for $i,j>N$, $x_i-x_j\in\mathfrak{a}^n$. Therefore $\{x_0,x_1,\cdots\}$ is a Cauchy sequence as desired.

ADDED IN PROOF. The author recently discovered that Theorem 2 can be derived from Taylor's lemma, see Y. Nouaze and P. Gabriel's *Idéaux premiers de l'algèbre enveloppante d'une algèbre de Lie nilpotente*, J. Algebra **6** (1967), 77–99.

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