A NOTE ON A LEMMA OF ZARISKI AND HIGHER DERIVATIONS

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Abstract. A sufficient condition is given for an a-adic complete ring R to be a power series ring over a subring.

1. Introduction. We prove in this note the following theorem: Let R be a ring and let a be an ideal in R such that \( \cap_{i=0}^\infty a^i = 0 \) and R is complete with respect to the a-adic topology. Assume that there exists a higher derivation \( D = \{D_i\}_{i=0}^\infty \) of R such that \( D_1(x) = 1 \) for some \( x \in a \). Let \( E = D_0 - x D_1 + \cdots + (-1)^n x^n D_n + \cdots \). If \( E(x) = 0 \), then there exists a subring \( R_1 \) of R such that \( R = R_1[[x]] \), and x is analytically independent over \( R_1 \).

This result generalizes Zariski's original lemma [5, Lemma 4, p. 526], and [1, Theorem 6, p. 412], a version of Zariski's lemma when R is of positive characteristic, and also removes the condition that R is an integral domain as we mentioned at the end of [1, p. 414]. [5, Lemma 4, p. 256] played a very important role in the study of analytic product of an affine algebraic variety \( V \) along a given subvariety \( W \) of \( V \) in A. Seidenberg's paper on differential ideals [4].

In the last section, we generalize a lemma of M. Miyanishi [2, p. 194] slightly, and give some remarks on his proofs of his lemma and Proposition 1.3 [2, p. 194].

2. Preliminaries. Throughout this note, all rings are commutative with identity. A derivation \( D \) of a ring R is an additive group homomorphism from R to R such that \( D(a \cdot b) = a Db + b Da \) for all a and b in R. A higher derivation \( D = \{D_i\}_{i=0}^\infty \) of a ring R is a sequence of additive group homomorphisms from R to R such that

1. \( D_0 \) = identity map on R, and
2. \( D_n(a \cdot b) = \sum_{i=0}^n D_i(a) \cdot D_{n-i}(b) \) for all \( n = 1, 2, \cdots \), and for all a and b in R. (Note \( D_1 \) is always a derivation of R.) Leibniz formula.

Let \( R \) be a ring and let \( a \) be an ideal in \( R \). Then \( R \) has a topological structure with \( \{a^i\}_{i=0}^\infty \) as the fundamental system of neighborhoods of the
zero in $R$. This is the so-called $a$-adic topology in $R$. Both addition and multiplication are continuous in $a$-adic topology. $R$ is a Hausdorff space if and only if $\cap_{i=0}^{\infty} a^i = 0$. A mapping $f: R \rightarrow R$ is continuous if there exists a subsequence $\{a^n\}_{n=0}^{\infty}$ of the sequence $\{a^n\}_{n=0}^{\infty}$ such that $f(a^n) \subseteq a^n$, for each natural number $n$.

**Lemma 1.** Let $R$ be a ring. Let $D = \{D_i\}_{i=0}^{\infty}$ be a higher derivation of $R$. Then

1. For $a_1, \ldots, a_n \in R$, and for each natural number $m$,

$$D_m(a_1 \cdots a_n) = \sum_{i=1}^{n} a_1 \cdots \hat{a}_i \cdots a_n D_m(a_i)$$

$$+ \sum_{i=1}^{m-1} D_i(a_1 \cdots a_{n-1}) \cdot D_{m-i}(a_n)$$

$$+ \sum_{i=1}^{m-1} \sum_{j=2}^{n-1} a_{j+1} \cdots a_n D_i(a_1 \cdots a_{j-1}) D_{m-i}(a_j),$$

where $a_1 \cdots \hat{a}_i \cdots a_n = a_1 \cdots a_{i-1} a_{i+1} \cdots a_n$.

2. For each ideal $a$ in $R$, and for each natural number $m$, $D_m(a^n) \subseteq a^{n-m}$ for each $a^n$, i.e. $D_m$ is continuous with respect to the $a$-adic topology.

3. For each ideal $a$ in $R$ and for each $x \in a$, let $E_x = \sum_{i=0}^{\infty} (1)^i x^i D_i$. For each natural number $l$, $E_l(a^n) \subseteq a^n$ for each $a^n$, i.e. $E_l$ is a continuous additive group homomorphism of $R$ with respect to the $a$-adic topology.

**Proof.** Straightforward.

**Lemma 2.** Let $R$ be a ring. Let $a$ be an ideal of $R$ such that $\cap_{i=0}^{\infty} a^i = 0$. Assume $R$ is complete with respect to the $a$-adic topology. Let $x \in a$, and let $E_x = \sum_{i=0}^{\infty} (-1)^i x^i D_i$. Then the sequence $\{E_x\}_{i=0}^{\infty}$ is uniformly convergent, and $E = \sum_{i=0}^{\infty} (-1)^i x^i D_i = \lim_{i \to \infty} E_i$ is a continuous endomorphism of $R$.

**Proof.** The sequence $\{E_x(a)\}$ is a Cauchy sequence for each $a \in R$. In fact $E_x(a) - E_x(a) \in a^n$ for $i, j > n$; i.e., for each given $a^n$, there exists a natural number $N (= n)$ such that $E_x(a) - E_x(a) \in a^n$ for $i, j > N$. Hence $\{E_x(a)\}$ converges in $R$. Since the natural number $N$ above is independent of $a \in R$, therefore $\{E_x\}_{i=0}^{\infty}$ converges uniformly in $R$. Hence $E = \lim_{i \to \infty} E_i$ is continuous. In fact for each $a^n$ and for each $a \in a^n$, there exists a natural number $N = n$ which is independent of $a$ such that $E(a) = (E(a) - E_x(a)) + E_x(a) \in a^n$. Thus $E(a^n) \subseteq a^n$.

Since $E(a + b) = \lim_{i \to \infty} E_i(a + b) = \lim_{i \to \infty} E_i(a) + \lim_{i \to \infty} E_i(b) = E(a) + E(b)$, therefore $E(a + b) = E(a) + E(b)$. Also for each natural number $l$,
$E_i(a) \cdot E_i(b) \equiv E_i(ab) \mod(x^{i+1})$. Thus $E_i(a)E_i(b) - E_i(ab) \in a^{i+1}$. Therefore

$$\lim_{i \to \infty} E_i(a) \cdot E_i(b) = \lim_{i \to \infty} E_i(ab).$$

But $\lim_{i \to \infty} E_i(a) \cdot E_i(b) = \lim_{i \to \infty} E_i(a) \cdot \lim_{i \to \infty} E_i(b)$. So $E(ab) = E(a)E(b)$.

3. Theorem 1. Let $R$ be a ring and let $a$ be a proper ideal in $R$ such that $\bigcap_{n=0}^{\infty} a^n = 0$, and $R$ is complete with respect to the $a$-adic topology. Assume that there exists a higher derivation $D = \{D_i\}_{i=0}^{\infty}$ of $R$ such that $D_1(x) = 1$ for some $x \in a$. Let $E = D_0 - xD_1 + \cdots + (-1)^n x^n D_n + \cdots$. If $E(x) = 0$, then there exists a subring $R_1$ of $R$ such that $R = R_1[[x]]$, and $x$ is analytically independent over $R_1$; i.e. if $\sum_{n=0}^{\infty} a_i x^n = 0$ where $a_i \in R_1$ then

$$a_i = 0 \text{ for all } i = 1, 2, \ldots .$$

Proof. $E(x) = 0$ implies $E^{-1}(0) = R_x$. Indeed, if $a \in E^{-1}(0)$, then $E(a) = 0$; i.e. $a - xD_1 a + \cdots + (-1)^n x^n D_n a + \cdots = 0$. Thus $a = xD_1 a + \cdots + (-1)^n x^n D_n a + \cdots$ is in $R_x$. So $E^{-1}(0) \subseteq R_x$. The other inclusion is obvious. Next, we observe that $E^2 = E$. In fact for each $a \in R$,

$$E(a) = a - xD_1 a + \cdots$$

and

$$E^2(a) = E(a) - E(xD_1 a + \cdots) = E(a) - E(x) \cdot E(D_1 a - xD_2 a + \cdots) = E(a),$$

so $E^2(a) = E(a)$ for all $a \in R$. Let $R_1 = E(R)$ then $E$ is an identity map on $R_1$. Let $a$ be an arbitrary element in $R$. $E(a) = a - xD_1 a + \cdots + (-1)^n x^n D_n a + \cdots$ implies that $a = E(a) + a_1 x$ for some $a_1 \in R$. Thus $a = E(a) + xE(a_1) + a_2 x^2$ for some $a_2 \in R$, and so on. Therefore we have $a = E(a) + xE(a_1) + \cdots + x^n E(a_n) + \cdots$ for some $a_1, a_2, \ldots, a_n, \ldots$ in $R$, and $a \in R_1[[x]]$. Hence $R = R_1[[x]]$. Finally we suppose $b_0 + b_1 x + \cdots + b_n x^n + \cdots = 0$ where $b_0, b_1, \ldots, b_n, \ldots$ are in $R_1$. We prove inductively $b_0 = b_1 = \cdots = b_n = \cdots = 0$. Since $E$ is identity on $R_1$, $E(b_0 + b_1 x + \cdots + b_n x^n + \cdots) = 0$ implies $b_0 = E(b_0) = 0$. Assume $b_0 = b_1 = \cdots = b_i = 0$. We have $b_{i+1} x^{i+1} + b_{i+2} x^{i+2} + \cdots = 0$. By Lemma 1 and $D_1(x) = 1$, we have $D_n x^n \equiv 0 \mod(x)$ and $D_n x^{n+j} \equiv 0 \mod(x)$ for all natural numbers $n$ and $j$. Thus

$$0 = D_{i+1}(b_{i+1} x^{i+1} + b_{i+2} x^{i+2} + \cdots)$$

$$= D_{i+1}(b_{i+1} x^{i+1}) + D_{i+1}(b_{i+2} x^{i+2} + \cdots)$$

$$= b_{i+1} D_{i+1}(x^{i+1}) \mod(x) = b_{i+1} \mod(x).$$

Therefore $b_{i+1} + c_{i+1} x = 0$ for some $c_{i+1} \in R$. So $0 = E(b_{i+1} + c_{i+1} x) = b_{i+1}$.

Therefore $x$ is analytically independent over $R_1$. 

If $R$ contains the field of rational numbers as a subring, then every derivation $D$ of $R$ gives rise to a higher derivation of $R$, namely, \( \{D_0, D, D^2/2!, \ldots, D^n/n!, \ldots\} \) where $D^n$ is the $n$th successive derivation of $D$, and $D_0$ is the identity map in $R$. Let $a$ be an ideal of $R$ such that $R$ is a complete Hausdorff space with respect to the $a$-adic topology. Assume $Dx = 1$ for some $x \in a$. Then the endomorphism $E = e^{-xD} = \sum (-1)^n x^n D^n/n!$ always maps $x$ to zero. Let $R_1 = E(R)$, then $D(R_1) = 0$. Thus we have the following

**Corollary 1.** Let $R$ be a ring containing the field of rational numbers as a subring. Let $a$ be an ideal in $R$ such that $R$ is a complete Hausdorff space with respect to the $a$-adic topology. Assume there exists a derivation $D$ of $R$ such that $Dx = 1$ for some $x \in a$. Then there exists a subring $R_1$ of $R$ such that (1) $D$ is zero on $R_1$ and (2) $R = R_1[[x]]$ and $x$ is analytically independent over $R_1$.

In the following, a semilocal (local) ring $\mathfrak{D}$ is a Noetherian ring with finitely many (unique) maximal ideals. Let $m$ be the intersection of the maximal ideals of $\mathfrak{D}$. It is well known that $\bigcap_{i=0}^{\infty} m^i = 0$. In this case we use $m$-adic topology for $R$. As a corollary to Corollary 1, we have the original lemma of Zariski.

**Corollary 2.** Let $(\mathfrak{D}, m)$ be a complete semilocal ring of characteristic zero. Let $D$ be a derivation of $\mathfrak{D}$. Assume that there exists an element $x$ in $m$ of $1)$ such that $Dx$ is a unit in $\mathfrak{D}$. Then $\mathfrak{D}$ contains a ring $\mathfrak{D}_1$ of representatives of the (complete) semilocal ring $\mathfrak{D}/\mathfrak{D}x$ having the following properties: (a) $D$ is zero on $\mathfrak{D}_1$; (b) $x$ is analytically independent on $\mathfrak{D}_1$; (c) $\mathfrak{D}$ is the power series ring $\mathfrak{D}_1[[x]]$.

**Proof.** Replace $D$ by $(1/ Dx)D$ and apply Corollary 1.

We also get [1, Theorem 6, p. 412] as a corollary to the theorem.

**Corollary 3.** Let $(\mathfrak{D}, m)$ be a complete local ring. Let $x \in m$ and let $D = \{D_i\}_{i=0}^{\infty}$ be a higher derivation of $\mathfrak{D}$ such that $D_1x$ is a unit in $\mathfrak{D}$, and $D_ix = 0$ for $i > 1$. Then there exists a subring $\mathfrak{D}_1$ of $\mathfrak{D}$ such that: (a) $\mathfrak{D}_1$ is a complete local ring, (b) $x$ is analytically independent over $\mathfrak{D}_1$, and (c) $\mathfrak{D} = \mathfrak{D}_1[[x]]$.

**Proof.** Let $D_1x = e^{-1}$, where $e$ is a unit in $\mathfrak{D}$. Replacing $\{D_i\}_{i=0}^{\infty}$ by $\{eD_i\}_{i=0}^{\infty}$, we may assume $D_1x = 1$. Since $D_ix = 0$ for $i > 1$, therefore $Ex = x - xD_1x = 0$ where $E = D_0 - xD_1 + \cdots + (-1)^n x^n D_n + \cdots$. Thus the theorem is applicable.

**Remarks.** (1) Theorem 1 and Corollary 1 hold under the assumption that $D_1x$ is a unit. The proofs are easily modified.
(2) If \( R \) has \( \mathfrak{a} \) as its sole maximal ideal and is a complete Hausdorff space with respect to the \( \mathfrak{a} \)-adic topology, then \( E(x) = 0 \) and \( x \neq 0 \) implies that \( D_1(x) \) is a unit.

(3) If \( R \) is a complete Hausdorff integral domain with respect to the \( \mathfrak{a} \)-adic topology, then \( E(x) = 0 \) and \( x \neq 0 \) implies that \( D_1(x) \) is a unit.

4. Though the following theorem could be easily proved by a similar technique used in the proof of Theorem 1, we would like to prove it as a corollary to Corollary 1.

**Theorem 2.** Let \( R \) be a ring containing the field of rational numbers as a subring. Let \( \mathfrak{a} \) be an ideal in \( R \) such that \( \cap \mathfrak{a}^i = 0 \). Assume that there exists a derivation \( D \) of \( R \) such that (1) for each \( \gamma \in R \), \( D(\gamma) = 0 \) for sufficiently large \( i \), and (2) \( D(x) = 1 \) for some \( x \in \mathfrak{a} \). Then (a) there exists a subring \( R_1 \) of \( R \) such that \( R = R_1[x] \) and \( x \) is algebraically independent over \( R_1 \); (b) \( D \) is trivial on \( R_1 \).

**Proof.** Let \( \hat{R} \) be the completion of \( R \) with respect to the \( \mathfrak{a} \)-adic topology. Then \( \hat{R} \) is a complete Hausdorff space with respect to the topology defined by the filtration \( \mathfrak{a} \supseteq (\mathfrak{a}^2) \supseteq \cdots \supseteq (\mathfrak{a}^n) \supseteq \cdots \), where \( (\mathfrak{a}^i)^{\circ} \) is the closure of \( \mathfrak{a}^i \) in \( \hat{R} \). Thus \( \bigcap_{i=0}^{\infty} (\mathfrak{a}^i)^{\circ} = 0 \). Note that \( (\mathfrak{a}^i)^{\circ} = \mathfrak{a}^i \hat{R} \) in general and equality holds if \( \mathfrak{a} \) is finitely generated. Let \( \gamma_1, \gamma_2, \cdots, \gamma_n, \cdots \) be a Cauchy sequence in \( R \); Lemma 1 implies that \( \{D^i(\gamma_j)\}_{i=0}^{\infty} \) is also a Cauchy sequence for each \( j \). Define \( \hat{D}^i(\gamma) = \lim_{i \to \infty} D^i(\gamma_j) \). Then it is easy to check that \( \{\hat{D}_0, \hat{D}_1, \cdots, \hat{D}^n/n!, \cdots\} \) forms a higher derivation in \( \hat{R} \). Moreover \( \hat{D}^i(\mathfrak{a}^j) \subseteq (\mathfrak{a}^{-i})^{\circ} \). Indeed let \( \hat{\gamma} \in (\mathfrak{a}^j)^{\circ} \) and let \( \gamma_1, \cdots, \gamma_n, \cdots \) be a sequence in \( \mathfrak{a}^j \) such that \( \lim_{n \to \infty} \gamma_n = \hat{\gamma} \). Then \( \hat{D}^i(\hat{\gamma}) = \lim_{n \to \infty} D^i(\gamma_n) \). Since \( D^i(\gamma_n) \in \mathfrak{a}^{-i} \), therefore \( \hat{D}^i(\hat{\gamma}) \in (\mathfrak{a}^{-i})^{\circ} \). Lemma 1 and Lemma 2 are easily verified. Since the kernel of the natural ring homomorphism from \( R \) to \( \hat{R} \) is \( \mathfrak{a} = 0 \), \( R \) is viewed as a subring of \( \hat{R} \). \( \hat{D}^i \) restricted to \( R \) is \( D^i \) so \( \hat{D}^i x = 1 \) and

\[
\hat{E}(x) = x - (x/1!) \hat{D}(x) + \cdots + (-1)^n(x^n/n!) \hat{D}^n(x) + \cdots
\]

Thus it follows from Theorem 1 that \( \hat{R} = \hat{R}_1[[x]] \), where \( \hat{R}_1 \) is a subring of \( \hat{R} \) and \( x \) is analytically independent over \( \hat{R}_1 \). Let \( R_1 = R \cap \hat{R}_1 \). Then \( R = R_1[x] \). Indeed, let \( \gamma \in R \). Then \( \gamma = a_0 + a_1 x + \cdots + a_n x^n + \cdots \) where \( a_i \in \hat{R}_1 \) for all \( a_i \). Since there exists a natural number \( N \) such that \( D^N(\gamma) = 0 \). Therefore

\[
0 = \hat{D}^N(\gamma) = N! a_N + \frac{1}{2}(N + 1)! a_{N+1} x + \cdots.
\]

Hence \( a_i = 0 \) for all \( i \geq N \), and \( \gamma = a_0 + a_1 x + \cdots + a_{N-1} x^{N-1} \). It follows that \( R \subseteq \hat{R}_1[x] \). Applying \( D^{N-1} \) to \( \gamma \), we have \( D^{N-1}(\gamma) = (N-1)! a_{N-1} \).
Therefore \( a_{N-1} \in R \). Applying \( D^{N-2} \) to \((y-a_{N-1}x^{N-1})=a_0+a_1x+\cdots+a_{N-2}x^{N-2}\), we get \( a_{N-2} \in R \) and so on. Consequently, \( a_0, \cdots, a_{N-1} \) are all in \( R_1 \cap R=R_1 \). So \( R=R_1[x] \), and \( x \) is of course algebraically independent over \( R_1 \).

We would like to thank Professor M. Miyanishi for communicating to us the following result which we also observed independently.

**Proposition 1.** Let \( R \) be an integral domain of characteristic 0. Assume there is a derivation of \( R \) such that \( D^i(a)=0 \) for each \( a \in R \) and for sufficiently large \( i \). Then \( Dx=0 \) for all units \( x \) in \( R \).

**Proof.** Let \( x \) be a unit in \( R \), and let \( y \in R \) be such that \( xy=1 \). Then \( xDy+Dx=0 \). Suppose \( Dx \neq 0 \). Thus \( Dy \neq 0 \). Let \( i \) be the natural number such that \( D^ix=0 \) and \( D^mx \neq 0 \) for \( m<i \), also let \( j \) be the natural number such that \( D^jy=0 \) and \( D^py \neq 0 \) for \( p<j \). By Leibniz’s formula,

\[
0 = D^n(xy) = \sum_{k=0}^{n} \binom{n}{k} D^k(x)D^{n-k}(y).
\]

Taking \( n=i+j-2 \) and assuming \( i \leq j \) we get \( D^{i-1}(x) \cdot D^{i-1}(y)=0 \). Hence either \( D^{i-1}(x)=0 \) or \( D^{i-1}(y)=0 \), a contradiction.

Proposition 1 completes the proof of [2, Lemma 1.4, p. 194]. One could not use [2, Proposition 1.4, p. 194] to yield a proof to the last part of [2, Proposition 1.3, p. 193]. But Proposition 1 corrects that part of the proof.


**Theorem 3.** Let \( R \) be an integral domain of characteristic 0 with a unique maximal ideal \( m \) such that \( \cap_{i=0}^{\infty} m^i=0 \), i.e. \( (R, m) \) is a local domain which may not be Noetherian. If there is a derivation \( D \) of \( R \) such that \( D^i(a)=0 \) for each \( a \in R \) and for sufficiently large \( i \), then \( m \) is differential, i.e. \( D(m) \subseteq m \), and \( \bar{D} \) induced by \( D \) on \( R/m \) is trivial.

**Proof.** Suppose \( Dm \not\subseteq m \). Then there is \( x \in m \) such that \( Dx=u^{-1} \), where \( u \) is a unit in \( R \). Then \( uDx=1 \). Replacing \( D \) by \( uD \), we have \( (uD)^i(a)=uD^i(a) \) by Proposition 1. Thus \( (uD)^i(a)=0 \) for sufficiently large \( i \). It follows from Theorem 2 that \( R=R_1[x] \), a contradiction. The last part follows from Proposition 1.

Observing the fact that in a polynomial ring \( A[x] \) the units in \( A[X] \) are of the form \( a_0+a_1x+\cdots+a_nx^n \) such that \( a_0 \) is a unit in \( A \) and \( a_2, \cdots, a_n \) are nilpotent in \( A \), we give two examples countering Proposition 1 when \( R \) is not an integral domain.
Example 1. Let $R = \mathbb{Z}/(4)[X]$, where $\mathbb{Z}$ is the domain of integers and $X$ is an indeterminate over $\mathbb{Z}/(4)$. Let $D$ be a derivation of $R$ such that $DX = 1 + 2X + 2X^2$. $(DX)^2 = 1$, $D^2 X = 2 \neq 0$, $D^i(a) = 0$ for each $a \in R$ and for large integers $i$.

Example 2. Let $R = (\mathbb{Q}[t])[X]$, where $\mathbb{Q}$ is the field of rational numbers and $t$ is an indeterminate over $\mathbb{Q}[t]$. Let $D$ be a derivation of $R$ such that $DX = 1 + tX$. Then $DX$ is a unit ($(DX) \cdot (1-tX) = 1$, $D^2 X = t \neq 0$, and $Dt = 0, D^i(a) = 0$ for each $a \in R$ and for large $i$.

In the setting of Theorem 2, when $R$ is an integral domain, if there is a derivation $D$ such that $DX$ is a unit for some $x \in a$, then there is a $y \in a$ ($y = x/u$) such that $Dy = 1$. What can one say in a more general case? Both Example 1 and Example 2 give negative answers. Using an idea of Professor M. Rosenlicht [3, Theorem 1, p. 721] we prove the following theorem.

Theorem 4. Let $R$ be a ring, which contains the field of rational numbers, with an ideal $a$ such that $\bigcap_{i=0}^{\infty} a^i = 0$ and $R$ is complete with respect to the $a$-adic topology. Assume there is a derivation $D$ of $R$ such that $DX$ is a unit for some $x \in a$. Then there exists an element $y \in a$ such that $Dy = 1$.

Proof. If $DX$ is a unit then $D(x/Dx) - 1 \in R_x \subseteq a$. If we can construct a Cauchy sequence $\{x_0, x_1, \ldots, x_n, \ldots\}$ such that $x_i \in xR$ and $DX_i - 1 \in R_x^{i+1}$, then putting $y = \lim_{n \to \infty} x_i$, and since $a$ is also closed, we have $y \in a$ and $Dy = \lim_{n \to \infty} DX_i = 1$. The proposed construction goes inductively as follows: Since $D(R_x^{i+1}) \subseteq R_x^i$, $D$ induces a surjective $R$-homomorphism $\tilde{D}^i : R_x^{i+1} \to R_x^i \cap R_x^{i+1}$ such that $\tilde{D}^i(x^{i+1}) = (i + 1)x + R_x^{i+1}$.

Therefore there exists $z_i \in R_x^{i+1}$ such that

$$Dz_i \equiv DX_i - 1 \mod(R_x^{i+1})$$

for $i = 1, 2, \ldots$. Thus $D(x_i - z_i) - 1 \in R_x^{i+1}$. Putting $x_i = x_{i-1} - z_i$, we have a sequence $\{x_0 = x/Dx, x_1, x_2, \ldots\}$ such that $DX_i - 1 \in R_x^{i+1} \subseteq a^{i+1}$ for $i = 0, 1, 2, \ldots$. For a given $a^n$, there exists a positive integer $N (= n)$ such that for $i, j > N, x_i - x_j \in a^n$. Therefore $\{x_0, x_1, \ldots\}$ is a Cauchy sequence as desired.

Added in proof. The author recently discovered that Theorem 2 can be derived from Taylor’s lemma, see Y. Nouze and P. Gabriel’s *Idéaux premiers de l’algèbre enveloppante d’une algèbre de Lie nilpotente*, J. Algebra 6 (1967), 77–99.
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