PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 42, Number 2, February 1974

A NOTE ON MATRIX SOLUTIONS TO $A=XY-YX$

CHARLES R. JOHNSON[1]

ABSTRACT. It is known that a square matrix A can be written as a commutator $XY-YX$ if and only if $\mathrm{Tr}(A)=0$. In this note it is shown further that for a fixed A the spectrum of one of the factors may be taken to be arbitrary while the spectrum of the other factor is arbitrary as long as the characteristic roots are distinct. The distinctness restriction on one of the factors may not in general be relaxed.

I. **Introduction.** It was shown by Shoda [5] that for square matrices over a field of characteristic 0 the equation

$$(*) \qquad A = XY - YX$$

has a solution X, Y if and only if $\mathrm{Tr}(A)=0$. This was extended by Albert and Muckenhoupt [1] to arbitrary fields. In this note we show that with respect to their spectra solutions to (*) are nearly arbitrary. A recent result of Friedland [2] is employed and corollary facts about solutions to (*) are obtained.

II. **Results.** All matrices mentioned will be $n \times n$ matrices with entries from the complex field and will be denoted by capitals. Let $\sigma(A)$ denote the set of eigenvalues of A. Our main result is

THEOREM. *Let $\lambda_1, \cdots, \lambda_n, \cdots, \lambda_{2n}$ be arbitrary complex numbers except that $\lambda_i \neq \lambda_j$ if $i \neq j$ and $i, j \leq n$. Then if $\mathrm{Tr}(A)=0$ there is a solution X, Y to (*) with $\sigma(X)=\{\lambda_1, \cdots, \lambda_n\}$ and $\sigma(Y)=\{\lambda_{n+1}, \cdots, \lambda_{2n}\}$. Further, X may be chosen to be normal.*

Two lemmas which have appeared in the literature will be employed in the proof.

LEMMA 1 [3], [4]. *If $\mathrm{Tr}(A)=0$, then A is unitarily equivalent to a matrix $B=(b_{ij})$ with $b_{ii}=0$, $i=1, \cdots, n$.*

Received by the editors May 14, 1973.
AMS (MOS) subject classifications (1970). Primary 15A18, 15A24.
Key words and phrases. Commutator, trace, eigenvalues, positive definite.

[1] This work was done while the author was a National Academy of Sciences—National Research Council Postdoctoral Research Associate at the National Bureau of Standards, Washington, D.C. 20234.

© American Mathematical Society 1974

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

LEMMA 2 (FRIEDLAND [2]). *Let a_{ij}, $i\neq j$, $i,j=1,\cdots,n$, and $\alpha_1,\cdots,\alpha_n$ be prescribed elements from an algebraically closed field. If $A=(a_{ij})$, then the a_{ii}, $i=1,\cdots,n$, may be chosen so that $\sigma(A)=\{\alpha_1,\cdots,\alpha_n\}$.*

The first lemma is easily proven using induction and the fact that $\mathrm{Tr}(A)=0$ implies there is a nonzero column vector χ such that $\chi^*A\chi=0$. The proof of the second lemma is more involved and will not be repeated here.

It is worth noting that in the theorem the condition "$\lambda_i\neq\lambda_j$ if $i\neq j$ and $i,j\leqq n$" may not, in general, be relaxed.

EXAMPLE. Let $n=2$ and $A=\left(\begin{smallmatrix}0&\beta\\0&0\end{smallmatrix}\right)$, $\beta\neq 0$. Then $\mathrm{Tr}(A)=0$, but there is no pair X, Y such that $\sigma(X)=\sigma(Y)=\{0,0\}$ and $A=XY-YX$. Thus there is no pair X, Y such that $\sigma(X)=\{\alpha,\alpha\}$, $\sigma(Y)=\{\beta,\beta\}$ and $A=XY-YX$.

PROOF (OF THEOREM). Suppose $\mathrm{Tr}(A)=0$. By Lemma 1 there is a unitary matrix U such that $U^*AU=B=(b_{ij})$ with $b_{ii}=0$, $i=1,\cdots n$, and it is clear that the problem is unchanged under unitary equivalence since $B=RT-TR$ implies $A=(URU^*)(UTU^*)-(UTU^*)(URU^*)$.

Let $D=\mathrm{diag}\{\lambda_1,\cdots,\lambda_n\}$ and $c_{ij}=b_{ij}/(\lambda_i-\lambda_j)$, $i,j=1,\cdots,n$, $i\neq j$, and determine c_{ii}, $i=1,\cdots,n$, by Lemma 2, so that $\sigma(C)=\{\lambda_{n+1},\cdots,\lambda_{2n}\}$ where $C=(c_{ij})$.

Now $DC-CD=((\lambda_i-\lambda_j)c_{ij})=(b_{ij})$ and if $X=UDU^*$ and $Y=UCU^*$ we have $XY-YX=A$ and $\sigma(X)=\{\lambda_1,\cdots,\lambda_n\}$, $\sigma(Y)=\{\lambda_{n+1},\cdots,\lambda_{2n}\}$ as required. Since $X=UDU^*$, it is normal and the proof is complete.

COROLLARY. *If* $\mathrm{Tr}(A)=0$, *then A may be written $A=XY-YX$ where X and $Y+Y^*$ are Hermitian positive definite. If A is skew-Hermitian, X and Y may be taken Hermitian positive definite.*

PROOF. By the theorem choose X normal and λ_i real and positive, $i=1,\cdots,n$. This implies that X is Hermitian and positive definite. Pick Z so that $XZ-ZX=A$. It is clear that there is an $r>0$ such that $Y=Z+rI$ satisfies $Y+Y^*$ positive definite. Since $X(Z+rI)-(Z+rI)X=XZ-ZX$, we have $A=XY-YX$ as desired. If A is skew-Hermitian, choose X as before. It is easy to check that A is skew-Hermitian if and only if X and $Y-Y^*$ commute in (*). This means that $X((Y+Y^*)/2)-((Y+Y^*)/2)X=XY-YX$ and thus, in case A is skew-Hermitian, we may replace Y by $(Y+Y^*)/2$ which may be chosen positive definite Hermitian.

III. **Remarks.** Neither Lemma 1 nor Lemma 2 holds over arbitrary fields. Lemma 1 does hold over the real field **[4]**, but it is easy to construct examples to show that an algebraically closed field is not a superfluous assumption in Lemma 2. In particular Lemma 2 does not hold over the

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

real field. Thus further research would be of interest to see to what extent the theorem of this note holds over other fields.

References

1. A. A. Albert and B. Muckenhoupt, *On matrices of trace zero*, Michigan Math. J. **4** (1957), 1–3. MR **18**, 786.

2.[2] S. Friedland, *Matrices with prescribed off-diagonal elements*, Israel J. Math. **11** (1972), 184–189.

3. F. Gaines, *A note on matrices with zero trace*, Amer. Math. Monthly **73** (1966), 630–631. MR **33** #7356.

4. W. V. Parker, *Sets of complex numbers associated with a matrix*, Duke Math. J. **15** (1948), 711–715. MR **10**, 230.

5. K. Shoda, *Einige Sätze über Matrizen*, Japan J. Math. **13** (1936), 361–365.

National Bureau of Standards, Washington, D.C. 20234

[2] The author wishes to thank Professor Olga Taussky Todd for alerting him to reference [2] which greatly simplified the proof of the main theorem.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use