

ON RELATIVE  $\mathfrak{F}$ -NORMALIZERS IN FINITE  
 SOLVABLE GROUPS

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ABSTRACT. In the theory of saturated formations of finite solvable groups, certain subgroups, called  $\mathfrak{F}$ -reducers, have been considered. In this note it is pointed out that the work of an  $\mathfrak{F}$ -reducer  $R$  is done by certain subsets of  $R$ , subsets that can be considered generalizations of P. Hall's relative system normalizers to an arbitrary subgroup of a given finite solvable group.

In [3] C. Graddon defines the concept of the  $\mathfrak{F}$ -reducer of a subgroup of a finite solvable group, where  $\mathfrak{F}$  is a locally defined formation. A subgroup  $H$  is contained in its  $\mathfrak{F}$ -reducer, being equal to it if and only if  $H$  is " $\mathfrak{F}$ -abnormal". We point out below that much of the work of the  $\mathfrak{F}$ -reducer  $R$  of a subgroup  $H$  is done by particular subsets of  $R$ , whose intersections with  $H$  are  $\mathfrak{F}$ -normalizers of  $H$ . These subsets cover or avoid the  $H$ -composition factors of the whole group, so they generalize the relative system normalizers of P. Hall [5].

The basic information about formations,  $\mathfrak{F}$ -projectors,  $\mathfrak{F}$ -abnormality, Sylow systems, and  $\mathfrak{F}$ -normalizers may be found in [2], Carter and Hawkes [1], and Huppert [6]. All groups are finite and solvable. If  $\mathfrak{F}$  is a formation and  $G$  is a group the  $G_{\mathfrak{F}}$  denotes the  $\mathfrak{F}$ -residual of  $G$ , that is, the smallest normal subgroup of  $G$  with quotient in  $\mathfrak{F}$ . From now on  $\mathfrak{F}$  will denote a (not necessarily subgroup closed) locally defined formation with the fixed integrated local definition  $\{\mathfrak{F}(p)\}$ , such that  $\mathfrak{F}$  contains the nilpotent groups. A Sylow system  $\mathfrak{S}$  of a group  $G$  is a collection of Hall  $p'$ -subgroups  $S^p$  of  $G$ , one for each prime  $p$ .  $\mathfrak{S}$  reduces into a subgroup  $H$  of  $G$  if the collection  $\mathfrak{S} \cap H = \{S^p \cap H\}$  is a Sylow system of  $H$ .

Suppose that the Sylow system  $\mathfrak{S}$  reduces into the subgroup  $H$  of a group  $G$  then the  $\mathfrak{F}$ -reducer  $R_G(H, \mathfrak{F})$  of  $H$  is defined by Graddon in [3] to be the group generated by those elements  $x$  of  $G$  for which for all  $p(S^p)^x \cap H_{\mathfrak{F}(p)}$  is a Hall  $p'$ -subgroup of  $H_{\mathfrak{F}(p)}$ .

DEFINITION 1. The relative  $\mathfrak{F}$ -normalizer of  $\mathfrak{S}$  in  $G$  with respect to  $H$  is

$$D_G(\mathfrak{S}, H) = \{x \in G : (\forall p) S^p \cap H_{\mathfrak{F}(p)} = (S^p)^x \cap H_{\mathfrak{F}(p)}\},$$

where  $\mathfrak{S}$  reduces into  $H$ .

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The set  $D_G(\mathfrak{S}, H)$  is a group if  $H$  is  $\mathfrak{F}$ -abnormal in  $G$ , for then it becomes an  $\mathfrak{F}$ -normalizer of  $H$  (See Theorem 4 below), or if each  $H_{\mathfrak{F}(p)}$  is normal in  $G$ . In general  $D_G(\mathfrak{S}, H)$  need not be a group. (For suppose that  $\mathfrak{F}$  is the class of nilpotent groups,  $H \leq G$ ,  $H$  nilpotent then  $H$  is contained in the Fitting subgroup of  $D = D_G(\mathfrak{S}, H)$  if  $D$  is a group:  $H \leq D$  so assume  $G = D$ , then  $S^p \cap H \leq \bigcap_{x \in G} (S^p)^x \triangleleft G$ . Now consider Example 2.9 of Graddon [4]. In this  $G$  is generated by  $D_G(\mathfrak{S}, K)$  where  $K$  is a nilpotent subgroup that is not subnormal in  $G$ , thus  $G$  does not equal  $D_G(\mathfrak{S}, K)$ .)

PROPOSITION 2. *If  $\mathfrak{S}$  is a Sylow system of  $G$ , reducing into  $H \leq G$ , then*

$$R_G(H, \mathfrak{F}) = \langle H, D_G(\mathfrak{S}, H) \rangle.$$

PROOF. Clearly the left-hand side contains the right. Suppose  $x$  is a generator of  $R_G(H, \mathfrak{F})$  so for all  $p$   $S^p \cap H_{\mathfrak{F}(p)}$  and  $(S^p)^x \cap H_{\mathfrak{F}(p)}$  are Hall  $p'$ -subgroups of  $H_{\mathfrak{F}(p)}$ . Now  $\mathfrak{S} \cap H$  is a Sylow system of  $H$ , while there is a second Sylow system  $\mathfrak{X} = \{T^p\}$  of  $H$  such that

$$(\forall p)(S^p)^x \cap H_{\mathfrak{F}(p)} = T^p \cap H_{\mathfrak{F}(p)}.$$

By the conjugacy of Sylow systems there is an  $h$  in  $H$  with  $\mathfrak{X}^h = \mathfrak{S} \cap H$ . Thus  $xh$  is in  $D_G(\mathfrak{S}, H)$ , so completing the proof.

From Proposition 2 and Graddon's results in [3] one obtains that a subgroup  $H$  of  $G$  is  $\mathfrak{F}$ -abnormal if and only if  $H$  contains  $D_G(\mathfrak{S}, H)$ . We give a proof of this fact based on the following (characterizing) properties of  $\mathfrak{F}$ -abnormality, where  $H \leq G$ ,  $N \triangleleft G$ :

- (a) ([2, 1.2])  $H$   $\mathfrak{F}$ -abnormal in  $G$  implies  $HN/N$   $\mathfrak{F}$ -abnormal in  $G/N$ ,
- (b) ([2, 1.4])  $H$   $\mathfrak{F}$ -abnormal in  $HN$ ,  $HN/N$   $\mathfrak{F}$ -abnormal in  $G/N$  imply  $H$   $\mathfrak{F}$ -abnormal in  $G$ ,
- (c) ([1, 4.2]) if  $H$   $p$ -maximal in  $G$  then  $H$  is  $\mathfrak{F}$ -abnormal in  $G$  if and only if  $H \geq N_G(S^p \cap G_{\mathfrak{F}(p)})$  for some Hall  $p'$ -subgroup  $S^p$  of  $G$ .

LEMMA 3. *If  $H \leq G$ ,  $N \triangleleft G$ ,  $\mathfrak{S}$  a Sylow system of  $G$ , reducing into  $H$ , then*

$$D_G(\mathfrak{S}, H)N/N = D_{G/N}(\mathfrak{S}N/N, HN/N)$$

where  $\mathfrak{S}N/N = \{S^pN/N\}$ .

PROOF. The left-hand side is contained in the right, since a Hall subgroup remains a Hall subgroup upon taking quotients. Suppose  $xN$  is in the right-hand side, which gives  $(\forall p)$

$$S^pN/N \cap (HN/N)_{\mathfrak{F}(p)} = (S^pN/N)^x \cap (HN/N)_{\mathfrak{F}(p)}$$

and so

$$(S^p \cap H_{\mathfrak{F}(p)})N = ((S^p)^x \cap H_{\mathfrak{F}(p)})N.$$

We may assume that  $N$  is a  $q$ -group, for some prime  $q$ . If  $p=q$  then there is a  $y$  in  $N$  such that

$$S^p \cap H_{\mathfrak{F}(p)} = ((S^p)^x \cap H_{\mathfrak{F}(p)}N)^y \leq (S^p)^{xy} \cap H_{\mathfrak{F}(p)}$$

and, therefore, as the left-hand side is a Hall subgroup,  $S^p \cap H_{\mathfrak{F}(p)} = (S^p)^{xy} \cap H_{\mathfrak{F}(p)}$ . If  $p$  is not equal to  $q$  then

$$S^p \cap H_{\mathfrak{F}(p)} \leq (S^p)^x \cap H_{\mathfrak{F}(p)}N \cap H_{\mathfrak{F}(p)}$$

and so

$$S^p \cap H_{\mathfrak{F}(p)} = (S^p)^x \cap H_{\mathfrak{F}(p)} = (S^p)^{xy} \cap H_{\mathfrak{F}(p)}.$$

Thus  $xy \in D_G(\mathfrak{S}, H)$  and  $xN \in D_G(\mathfrak{S}, H)N/N$ .

**THEOREM 4.** *If  $H \leq G$  and  $\mathfrak{S}$  any Sylow system of  $G$  reducing into  $H$  then*

(a)  *$H$  is  $\mathfrak{F}$ -abnormal in  $G$  if and only if  $H \supseteq D_G(\mathfrak{S}, H)$ ,*

(b)  *$H$  is in  $\mathfrak{F}$  if and only if  $H \subseteq D_G(\mathfrak{S}, H)$ ,*

(c)  *$H$  is an  $\mathfrak{F}$ -projector of  $G$  if and only if  $H = D_G(\mathfrak{S}, H)$ .*

**PROOF.** (a) Suppose  $H$   $\mathfrak{F}$ -abnormal in  $G$  then by induction, if  $N$  is a minimal normal  $p$ -subgroup of  $G$ ,  $HN \supseteq D_G(\mathfrak{S}, H) = D_{HN}(\mathfrak{S} \cap HN, H)$  (as  $\mathfrak{S}$  reduces into  $HN$ ). Therefore it may be assumed that  $G = HN$  and  $H \cap N = 1$ . But now  $H_{\mathfrak{F}(p)}N = G_{\mathfrak{F}(p)}N$  and  $S^p \cap H_{\mathfrak{F}(p)} = S^p \cap G_{\mathfrak{F}(p)}$ , so by  $\mathfrak{F}$ -abnormality  $H \supseteq D_G(\mathfrak{S}, H)$ .

Conversely if  $H \supseteq D_G(\mathfrak{S}, H)$  then by induction  $HN/N$  is  $\mathfrak{F}$ -abnormal in  $G/N$ , for  $N$  minimal normal in  $G$ . Induction applies in  $HN$  so  $H$  is  $\mathfrak{F}$ -abnormal in  $HN$  and thus in  $G$  unless  $G = HN$  and  $H$  is  $p$ -maximal in  $G$  for some  $p$ . Now  $S^p \leq H$ ,  $D_G(\mathfrak{S}, H) \subseteq H$ , and again  $S^p \cap H_{\mathfrak{F}(p)} = S^p \cap G_{\mathfrak{F}(p)}$  so  $H \supseteq N_G(S^p \cap G_{\mathfrak{F}(p)})$ . Thus  $H$  is  $\mathfrak{F}$ -abnormal in  $G$ .

By definition  $H$  is in  $\mathfrak{F}$  if and only if  $(\forall p) S^p \cap H_{\mathfrak{F}(p)} \triangleleft H$ , so (b) is clear. Since  $H$  is an  $\mathfrak{F}$ -projector if and only if  $H$  in  $\mathfrak{F}$  and  $\mathfrak{F}$ -abnormal, (c) follows from (a) and (b).

Finally we have the covering and avoidance properties of the relative  $\mathfrak{F}$ -normalizers. Recall that if  $H \leq G$  then an  $H$ -composition factor  $L/K$  of  $G$  is  $\mathfrak{F}$ -central if  $H/C_H(L/K)$  is in  $\mathfrak{F}(p)$  where  $L/K$  is a  $p$ -group, otherwise it is  $\mathfrak{F}$ -eccentric. Equivalently  $L/K$  is  $\mathfrak{F}$ -central if and only if  $[L, S^p \cap H_{\mathfrak{F}(p)}] \leq K$  for some Hall  $p'$ -subgroup  $S^p$  of  $H$ . Now consider  $HL$ , for some  $H$ -composition factor  $L/K$  that is a  $p$ -group, and where  $L$  is taken to be subnormal in  $G$ . We have

$$HL \cap D_G(\mathfrak{S}, H) = D_{HL}(\mathfrak{S} \cap HL, H)$$

so for the present purposes assume  $G = HL$  and  $K = 1$ .

If  $L$  is  $\mathfrak{F}$ -central then  $L$  centralizes  $S^p \cap H_{\mathfrak{F}(p)}$ , so  $L \leq D_G(\mathfrak{S}, H)$ . If  $L$  is  $\mathfrak{F}$ -eccentric then  $H$  is  $\mathfrak{F}$ -abnormal in  $G$  (by [1, 2.3]). Now either  $H = G$

and  $D_G(\mathfrak{S}, G)$  becomes an  $\mathfrak{F}$ -normalizer of  $G$  thus (by [1, 4.1]) avoiding  $L$ , or  $H < G$  and by Theorem 4  $D_G(\mathfrak{S}, H)$  is contained in  $H$  so avoiding  $L$ .

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