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ON RELATIVE $\mathfrak{F}$-NORMALIZERS IN FINITE SOLVABLE GROUPS

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ABSTRACT. In the theory of saturated formations of finite solvable groups, certain subgroups, called $\mathfrak{F}$-reducers, have been considered. In this note it is pointed out that the work of an $\mathfrak{F}$-reducer R is done by certain subsets of R, subsets that can be considered generalizations of P. Hall's relative system normalizers to an arbitrary subgroup of a given finite solvable group.

In [3] C. Graddon defines the concept of the $\mathfrak{F}$-reducer of a subgroup of a finite solvable group, where $\mathfrak{F}$ is a locally defined formation. A subgroup H is contained in its $\mathfrak{F}$-reducer, being equal to it if and only if H is "$\mathfrak{F}$-abnormal". We point out below that much of the work of the $\mathfrak{F}$-reducer R of a subgroup H is done by particular subsets of R, whose intersections with H are $\mathfrak{F}$-normalizers of H. These subsets cover or avoid the H-composition factors of the whole group, so they generalize the relative system normalizers of P. Hall [5].

The basic information about formations, $\mathfrak{F}$-projectors, $\mathfrak{F}$-abnormality, Sylow systems, and $\mathfrak{F}$-normalizers may be found in [2], Carter and Hawkes [1], and Huppert [6]. All groups are finite and solvable. If $\mathfrak{F}$ is a formation and G is a group the $G_{\mathfrak{F}}$ denotes the $\mathfrak{F}$-residual of G, that is, the smallest normal subgroup of G with quotient in $\mathfrak{F}$. From now on $\mathfrak{F}$ will denote a (not necessarily subgroup closed) locally defined formation with the fixed integrated local definition $\{\mathfrak{F}(p)\}$, such that $\mathfrak{F}$ contains the nilpotent groups. A Sylow system $\mathfrak{S}$ of a group G is a collection of Hall p'-subgroups S^p of G, one for each prime p. $\mathfrak{S}$ reduces into a subgroup H of G if the collection $\mathfrak{S}\cap H=\{S^p\cap H\}$ is a Sylow system of H.

Suppose that the Sylow system $\mathfrak{S}$ reduces into the subgroup H of a group G then the $\mathfrak{F}$-reducer $R_G(H, \mathfrak{F})$ of H is defined by Graddon in [3] to be the group generated by those elements x of G for which for all $p(S^p)^x\cap H_{\mathfrak{F}(p)}$ is a Hall p'-subgroup of $H_{\mathfrak{F}(p)}$.

DEFINITION 1. The *relative $\mathfrak{F}$-normalizer* of $\mathfrak{S}$ in G with respect to H is

$$D_G(\mathfrak{S}, H) = \{x\in G: (\forall p) S^p \cap H_{\mathfrak{F}(p)} = (S^p)^x \cap H_{\mathfrak{F}(p)}\},$$

where $\mathfrak{S}$ reduces into H.

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The set $D_G(\mathfrak{S}, H)$ is a group if H is $\mathfrak{F}$-abnormal in G, for then it becomes an $\mathfrak{F}$-normalizer of H (See Theorem 4 below), or if each $H_{\mathfrak{F}(p)}$ is normal in G. In general $D_G(\mathfrak{S}, H)$ need not be a group. (For suppose that $\mathfrak{F}$ is the class of nilpotent groups, $H \leqq G$, H nilpotent then H is contained in the Fitting subgroup of $D = D_G(\mathfrak{S}, H)$ if D is a group: $H \leqq D$ so assume $G = D$, then $S^p \cap H \leqq \bigcap_{x \in G} (S^p)^x \lhd G$. Now consider Example 2.9 of Graddon [4]. In this G is generated by $D_G(\mathfrak{S}, K)$ where K is a nilpotent subgroup that is not subnormal in G, thus G does not equal $D_G(\mathfrak{S}, K)$.)

PROPOSITION 2. *If $\mathfrak{S}$ is a Sylow system of G, reducing into $H \leqq G$, then*

$$R_G(H, F) = \langle H, D_G(\mathfrak{S}, H)\rangle.$$

PROOF. Clearly the left-hand side contains the right. Suppose x is a generator of $R_G(H, \mathfrak{F})$ so for all p $S^p \cap H_{\mathfrak{F}(p)}$ and $(S^p)^x \cap H_{\mathfrak{F}(p)}$ are Hall p'-subgroups of $H_{\mathfrak{F}(p)}$. Now $\mathfrak{S} \cap H$ is a Sylow system of H, while there is a second Sylow system $\mathfrak{T} = \{T^p\}$ of H such that

$$(\forall p)(S^p)^x \cap H_{\mathfrak{F}(p)} = T^p \cap H_{\mathfrak{F}(p)}.$$

By the conjugacy of Sylow systems there is an h in H with $\mathfrak{T}^h = \mathfrak{S} \cap H$. Thus xh is in $D_G(\mathfrak{S}, H)$, so completing the proof.

From Proposition 2 and Graddon's results in [3] one obtains that a subgroup H of G is $\mathfrak{F}$-abnormal if and only if H contains $D_G(\mathfrak{S}, H)$. We give a proof of this fact based on the following (characterizing) properties of $\mathfrak{F}$-abnormality, where $H \leqq G$, $N \lhd G$:

(a) ([2, 1.2]) H $\mathfrak{F}$-abnormal in G implies HN/N $\mathfrak{F}$-abnormal in G/N,

(b) ([2, 1.4]) H $\mathfrak{F}$-abnormal in HN, HN/N $\mathfrak{F}$-abnormal in G/N imply H $\mathfrak{F}$-abnormal in G,

(c) ([1, 4.2]) if H p-maximal in G then H is $\mathfrak{F}$-abnormal in G if and only if $H \geqq N_G(S^p \cap G_{\mathfrak{F}(p)})$ for some Hall p'-subgroup S^p of G.

LEMMA 3. *If $H \leqq G$, $N \lhd G$, $\mathfrak{S}$ a Sylow system of G, reducing into H, then*

$$D_G(\mathfrak{S}, H)N/N = D_{G/N}(\mathfrak{S}N/N, HN/N)$$

where $\mathfrak{S}N/N = \{S^pN/N\}$.

PROOF. The left-hand side is contained in the right, since a Hall subgroup remains a Hall subgroup upon taking quotients. Suppose xN is in the right-hand side, which gives $(\forall p)$

$$S^pN/N \cap (HN/N)_{\mathfrak{F}(p)} = (S^pN/N)^x \cap (HN/N)_{\mathfrak{F}(p)}$$

and so

$$(S^p \cap H_{\mathfrak{F}(p)})N = ((S^p)^x \cap H_{\mathfrak{F}(p)}N)N.$$

We may assume that N is a q-group, for some prime q. If $p=q$ then there is a y in N such that

$$S^p \cap H_{\mathfrak{F}(p)} = ((S^p)^x \cap H_{\mathfrak{F}(p)}N)^y \leqq (S^p)^{xy} \cap H_{\mathfrak{F}(p)}$$

and, therefore, as the left-hand side is a Hall subgroup, $S^p \cap H_{\mathfrak{F}(p)} = (S^p)^{xy} \cap H_{\mathfrak{F}(p)}$. If p is not equal to q then

$$S^p \cap H_{\mathfrak{F}(p)} \leqq (S^p)^x \cap H_{\mathfrak{F}(p)}N \cap H_{\mathfrak{F}(p)}$$

and so

$$S^p \cap H_{\mathfrak{F}(p)} = (S^p)^x \cap H_{\mathfrak{F}(p)} = (S^p)^{xy} \cap H_{\mathfrak{F}(p)}.$$

Thus $xy \in D_G(\mathfrak{S}, H)$ and $xN \in D_G(\mathfrak{S}, H)N/N$.

THEOREM 4. *If $H \leqq G$ and $\mathfrak{S}$ any Sylow system of G reducing into H then*
(a) *H is $\mathfrak{F}$-abnormal in G if and only if $H \supseteq D_G(\mathfrak{S}, H)$,*
(b) *H is in $\mathfrak{F}$ if and only if $H \subseteq D_G(\mathfrak{S}, H)$,*
(c) *H is an $\mathfrak{F}$-projector of G if and only if $H = D_G(\mathfrak{S}, H)$.*

PROOF. (a) Suppose H $\mathfrak{F}$-abnormal in G then by induction, if N is a minimal normal p-subgroup of G, $HN \supseteq D_G(\mathfrak{S}, H) = D_{HN}(\mathfrak{S} \cap HN, H)$ (as $\mathfrak{S}$ reduces into HN). Therefore it may be assumed that $G=HN$ and $H \cap N = 1$. But now $H_{\mathfrak{F}(p)}N = G_{\mathfrak{F}(p)}N$ and $S^p \cap H_{\mathfrak{F}(p)} = S^p \cap G_{\mathfrak{F}(p)}$, so by $\mathfrak{F}$-abnormality $H \supseteq D_G(\mathfrak{S}, H)$.

Conversely if $H \supseteq D_G(\mathfrak{S}, H)$ then by induction HN/N is $\mathfrak{F}$-abnormal in G/N, for N minimal normal in G. Induction applies in HN so H is $\mathfrak{F}$-abnormal in HN and thus in G unless $G=HN$ and H is p-maximal in G for some p. Now $S^p \leqq H$, $D_G(\mathfrak{S}, H) \subseteq H$, and again $S^p \cap H_{\mathfrak{F}(p)} = S^p \cap G_{\mathfrak{F}(p)}$ so $H \geqq N_G(S^p \cap G_{\mathfrak{F}(p)})$. Thus H is $\mathfrak{F}$-abnormal in G.

By definition H is in $\mathfrak{F}$ if and only if $(\forall p)\ S^p \cap H_{\mathfrak{F}(p)} \lhd H$, so (b) is clear. Since H is an $\mathfrak{F}$-projector if and only if H in $\mathfrak{F}$ and $\mathfrak{F}$-abnormal, (c) follows from (a) and (b).

Finally we have the covering and avoidance properties of the relative $\mathfrak{F}$-normalizers. Recall that if $H \leqq G$ then an H-composition factor L/K of G is $\mathfrak{F}$-central if $H/C_H(L/K)$ is in $\mathfrak{F}(p)$ where L/K is a p-group, otherwise it is $\mathfrak{F}$-eccentric. Equivalently L/K is $\mathfrak{F}$-central if and only if $[L, S^p \cap H_{\mathfrak{F}(p)}] \leqq K$ for some Hall p'-subgroup S^p of H. Now consider HL, for some H-composition factor L/K that is a p-group, and where L is taken to be subnormal in G. We have

$$HL \cap D_G(\mathfrak{S}, H) = D_{HL}(\mathfrak{S} \cap HL, H)$$

so for the present purposes assume $G=HL$ and $K=1$.

If L is $\mathfrak{F}$-central then L centralizes $S^p \cap H_{\mathfrak{F}(p)}$, so $L \leqq D_G(\mathfrak{S}, H)$. If L is $\mathfrak{F}$-eccentric then H is $\mathfrak{F}$-abnormal in G (by [**1**, 2.3]). Now either $H=G$

and $D_G(\mathfrak{S}, G)$ becomes an $\mathfrak{F}$-normalizer of G thus (by [**1**, 4.1]) avoiding L, or $H<G$ and by Theorem 4 $D_G(\mathfrak{S}, H)$ is contained in H so avoiding L.

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